Game-Theoretic Models for Multi-Agent Interaction

Lasse Peters


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KORTE PRINSEN
GRACHT
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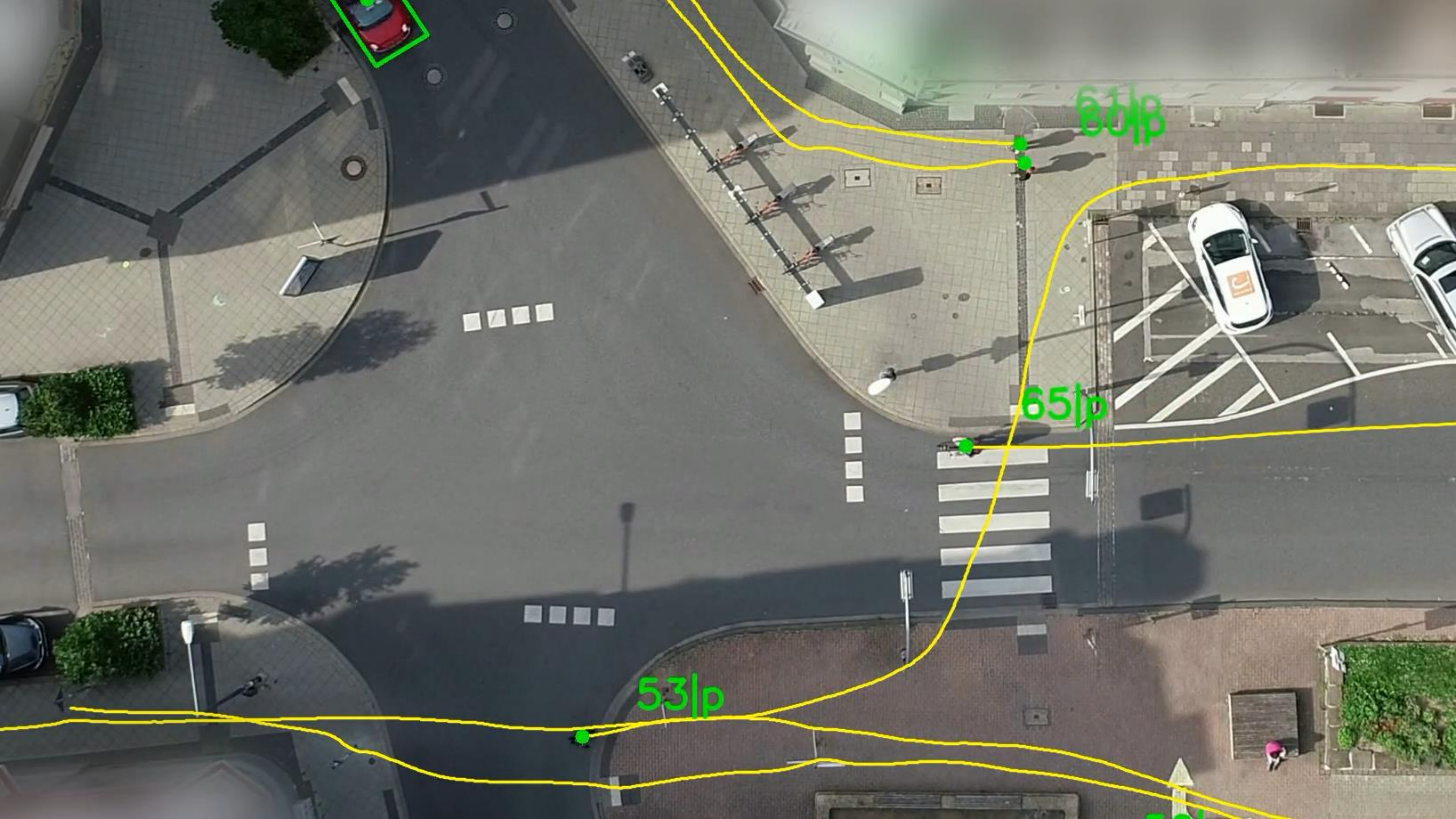
DBP
schlijper.nl
today


61|p
60|p
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CAR
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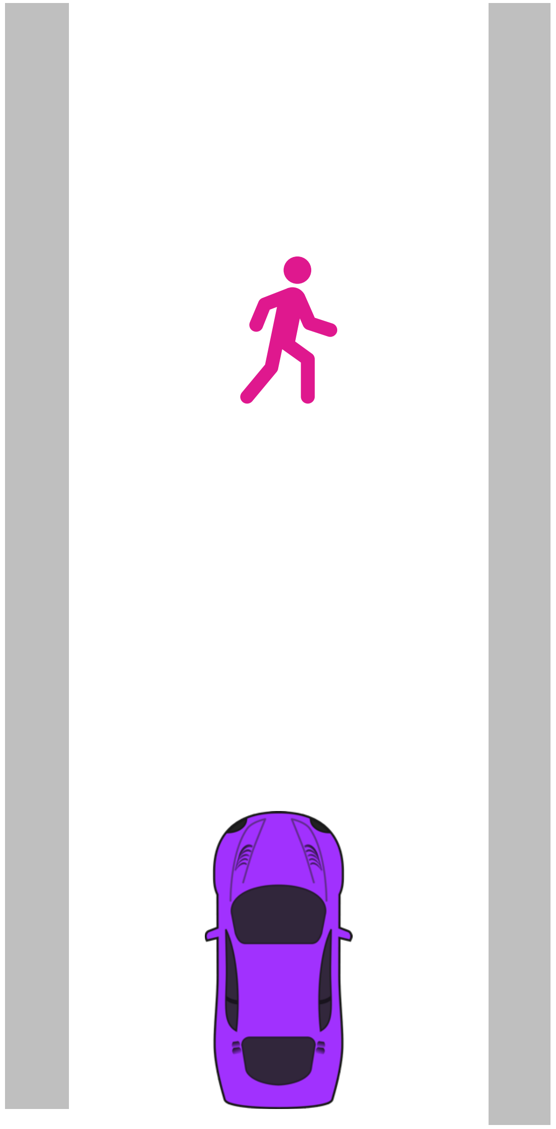
Interaction as a Game

$$\min_{\tau^i}$$

strategy

for now: sequence of controls

$i \in \{H, R\}$



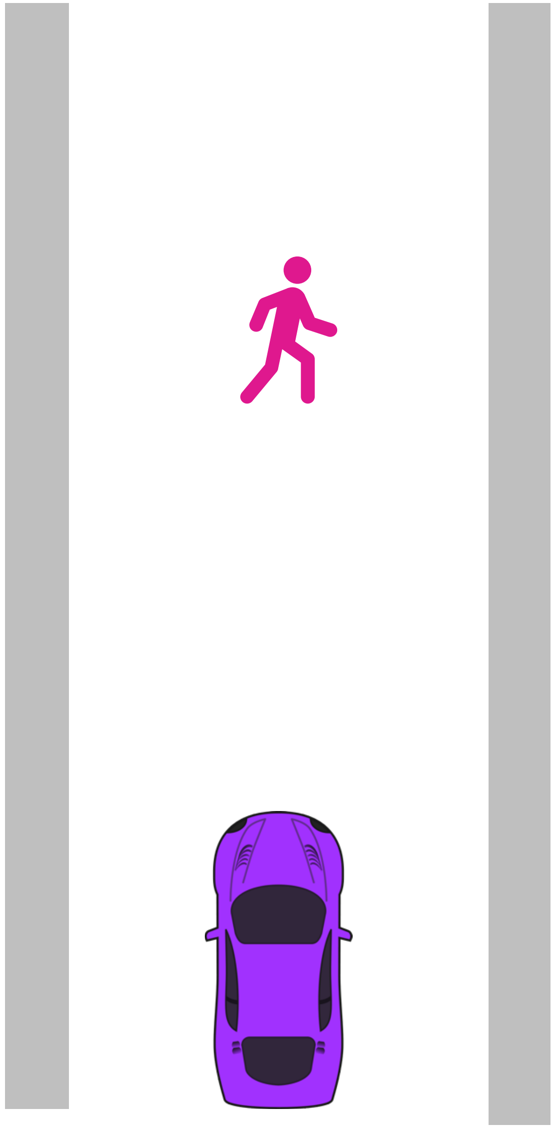
Interaction as a Game

*cost / **objective** function*

***other** agents' trajectory*

$$\min_{\tau^i} J^i(\tau^i, \tau^{\neg i})$$

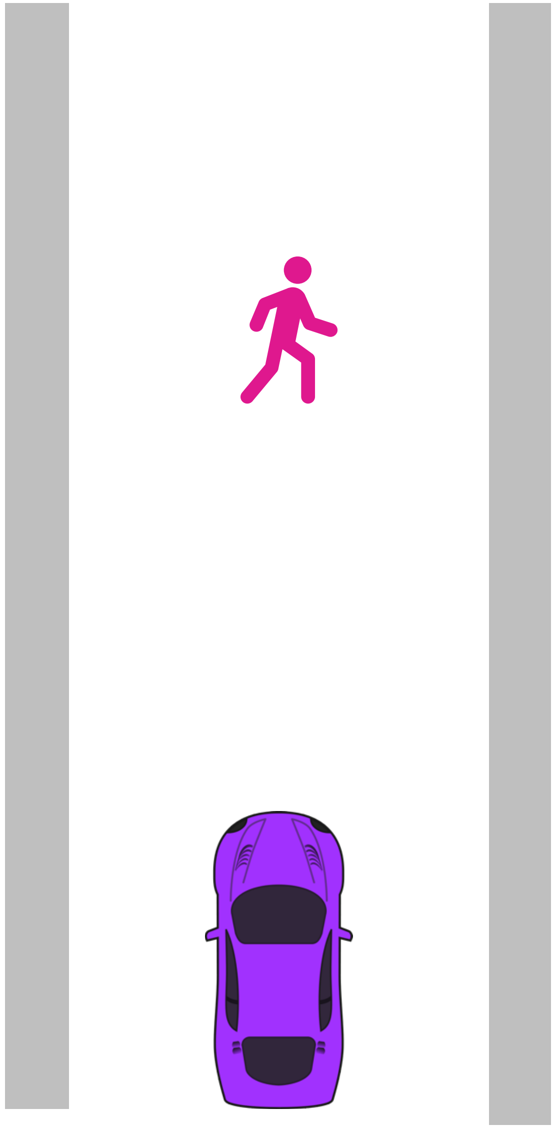
$$i \in \{H, R\}$$



Interaction as a Game

$$\left.\begin{aligned} &\min_{\tau^i} J^i(\tau^i, \tau^{\neg i}) \\ \text{s.t.}\ \ &\tau^i \in \mathcal{K}^i(\tau^{\neg i}) \end{aligned}\right\}\ i \in \{H, R\}$$

*Constraints (that depend on **other** agent(s))*



Interaction as a Game | Taxonomy

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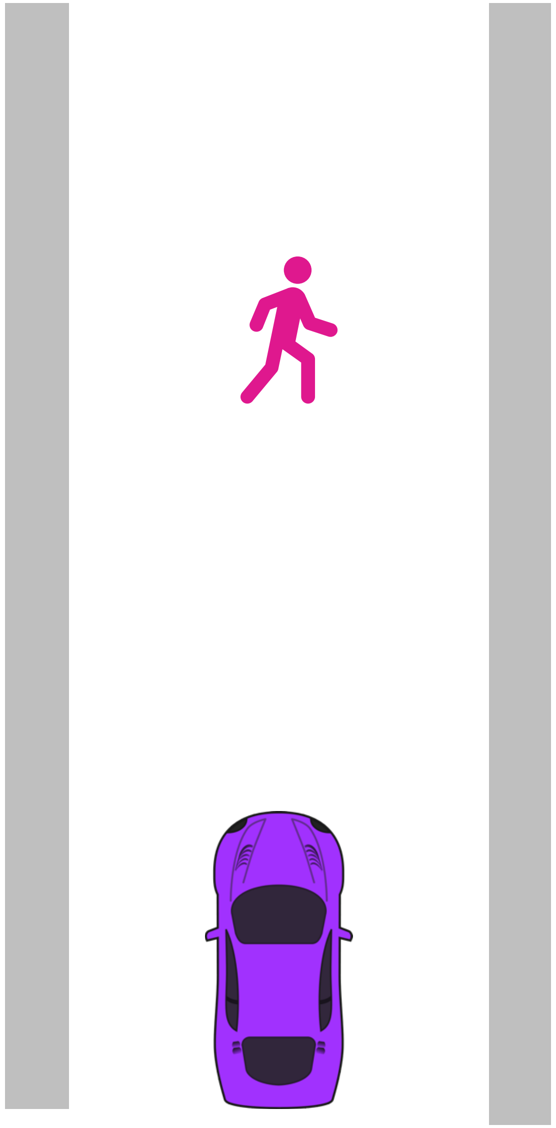
This talk:

- *discrete-time* systems (e.g. from direct transcription)
- *General-sum* cost structure



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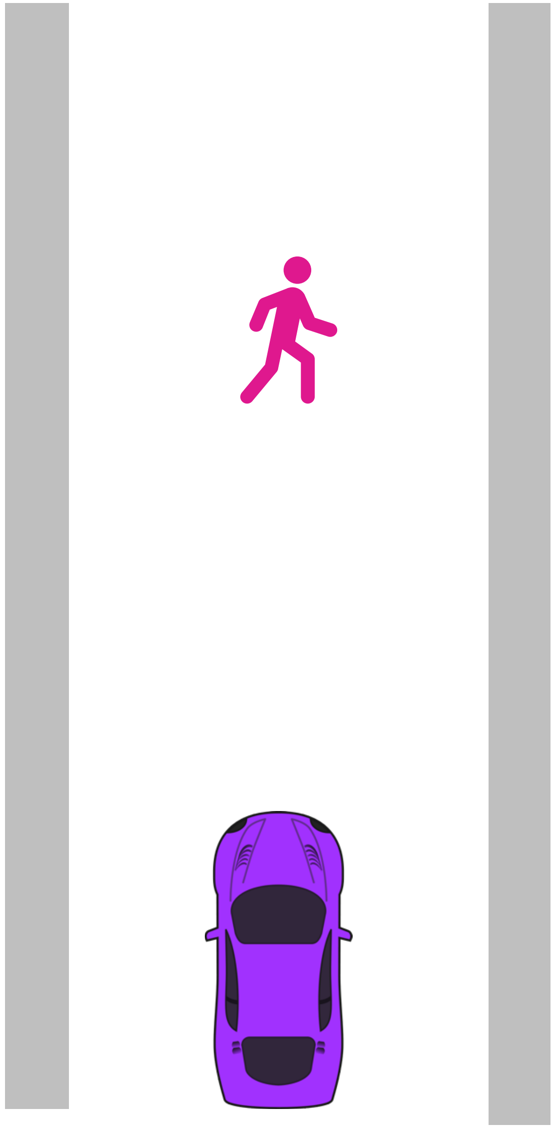

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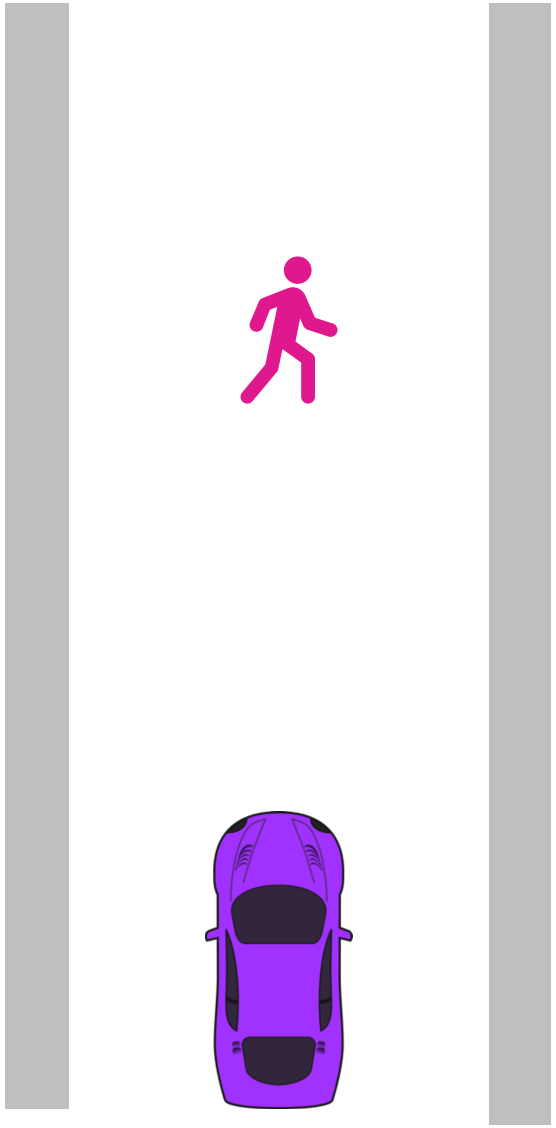

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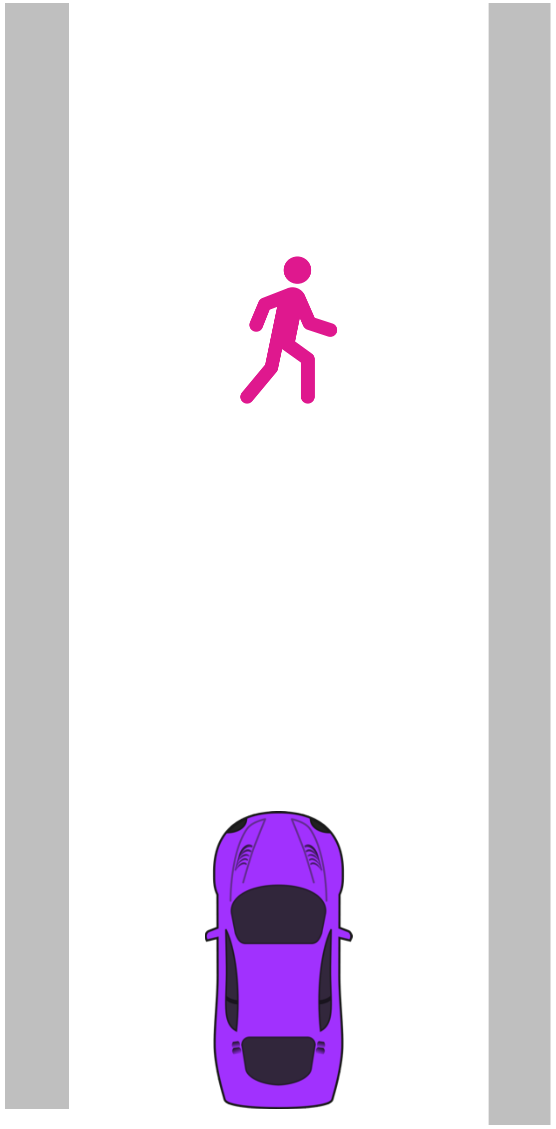
For now:

- *open-loop information structure*:
  τ^i is open-loop strategy
  (sequence of continuous control inputs)

Interaction as a Game

The **set of best responses** to opponent strategies $\tau^{\neg i}$

$$\left.\begin{aligned} \mathcal{S}^i\left(\tau^{\neg i}\right) \stackrel{\text{def}}{=} \arg\min_{\tau^i} J^i(\tau^i, \tau^{\neg i}) \\ \text{s.t. } \tau^i \in \mathcal{K}^i(\tau^{\neg i}) \end{aligned}\right\} \quad i \in \{H, R\}$$



Interaction as a Game

The **set of best responses** to opponent strategies τ^R for the *robot*

$$\mathcal{S}^R(\tau^H) \stackrel{\text{def}}{=} \arg\min_{\tau^R} J^R(\tau^R, \tau^H)$$

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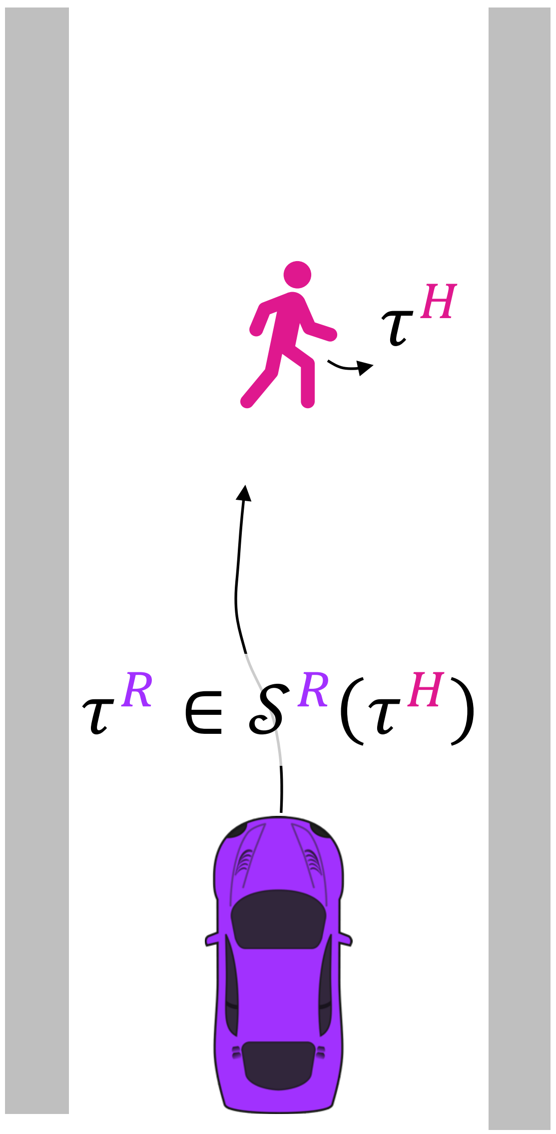

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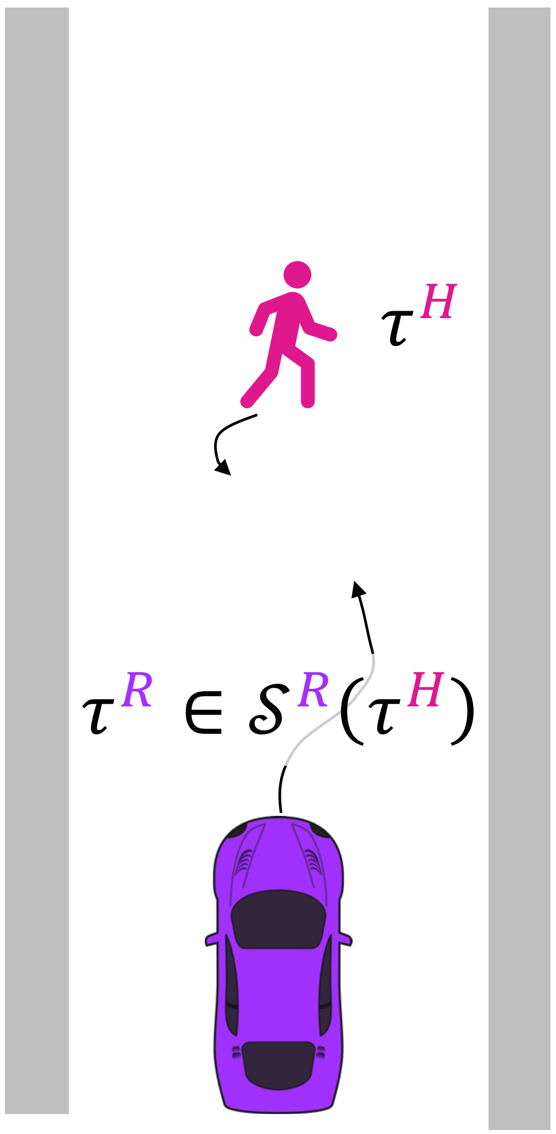

Interaction as a Game

The **set of best responses** to opponent strategies τ^R for the *human*

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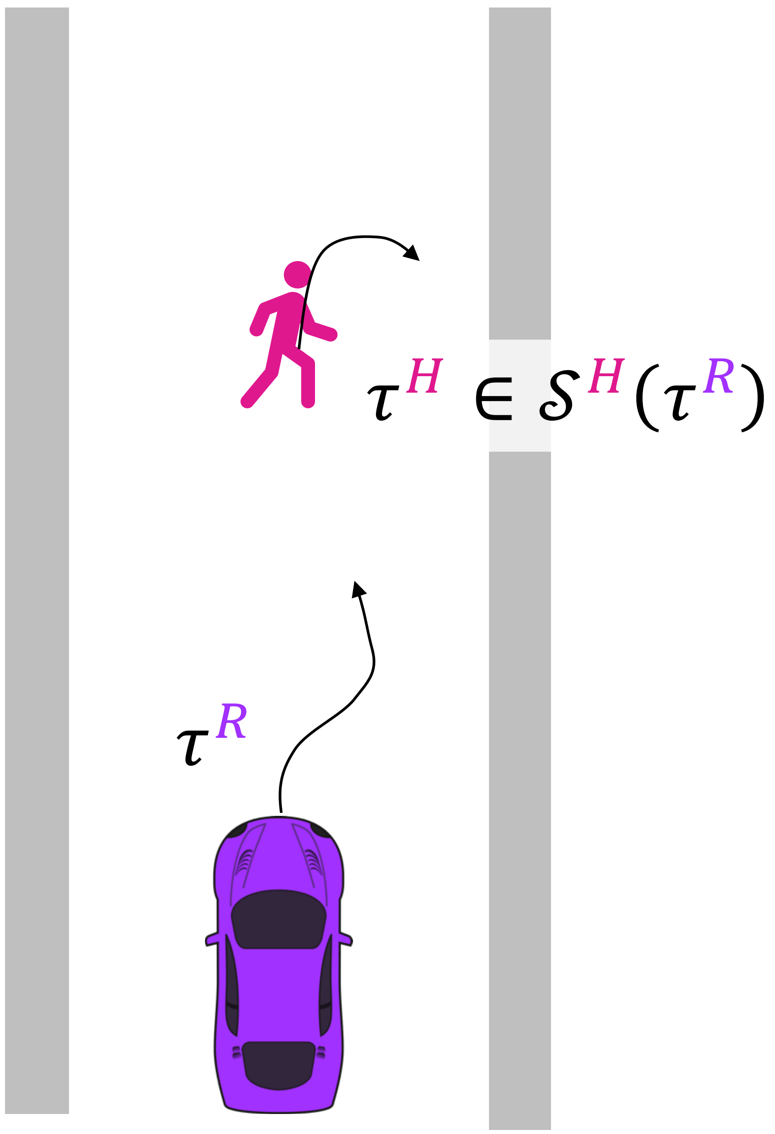

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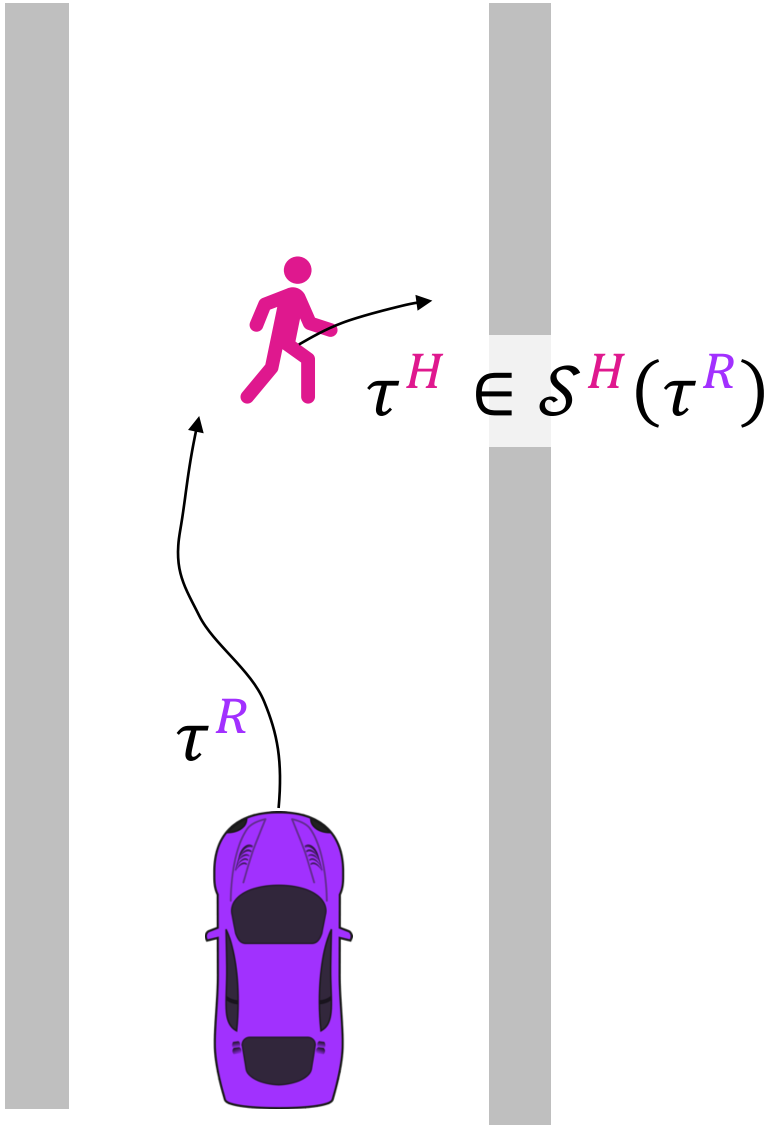

The Generalized Nash Equilibrium Concept

When **each player's strategy is a best response** to the others'

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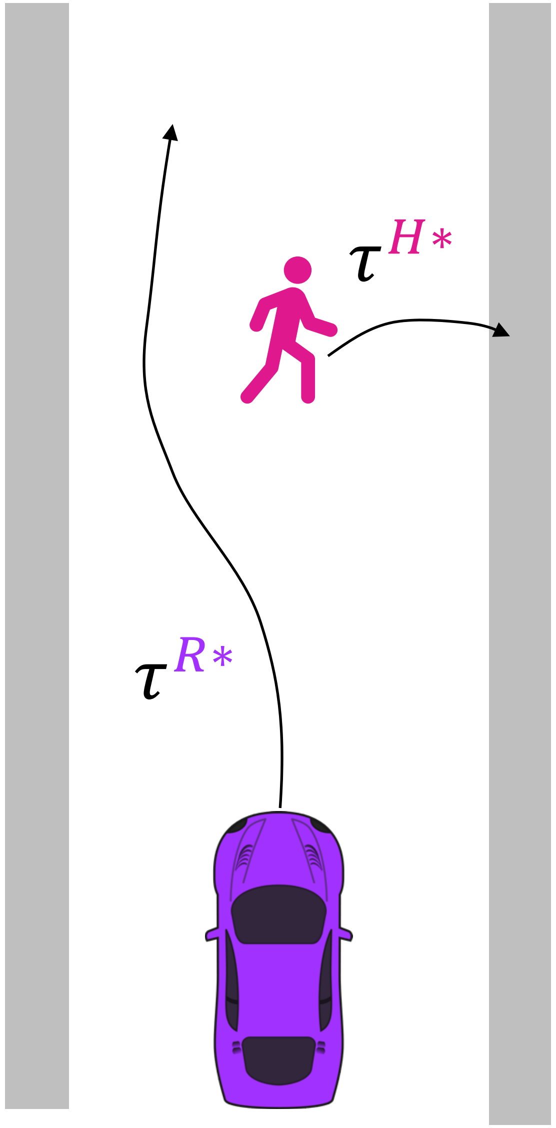

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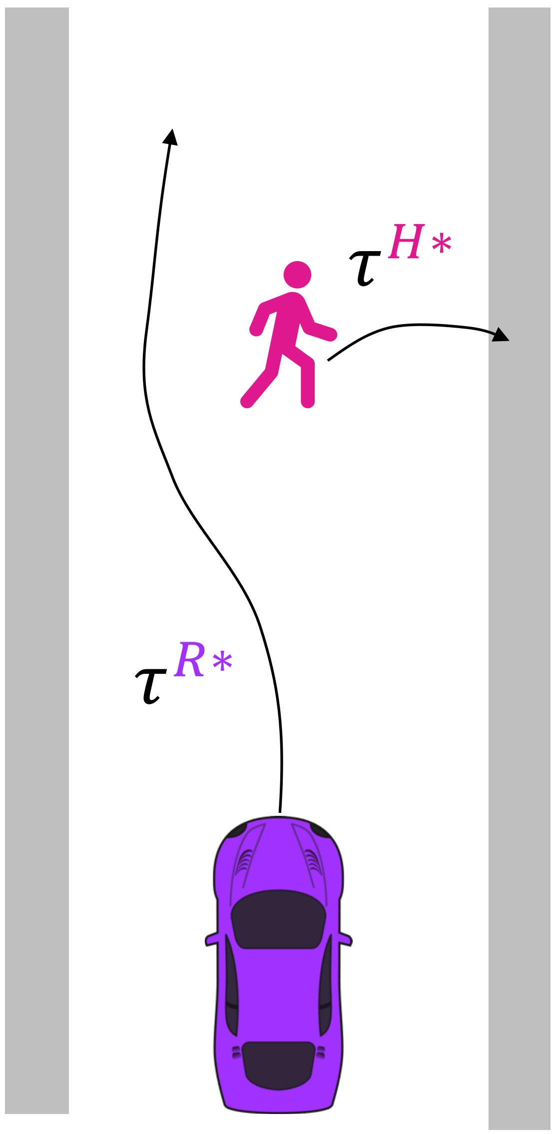

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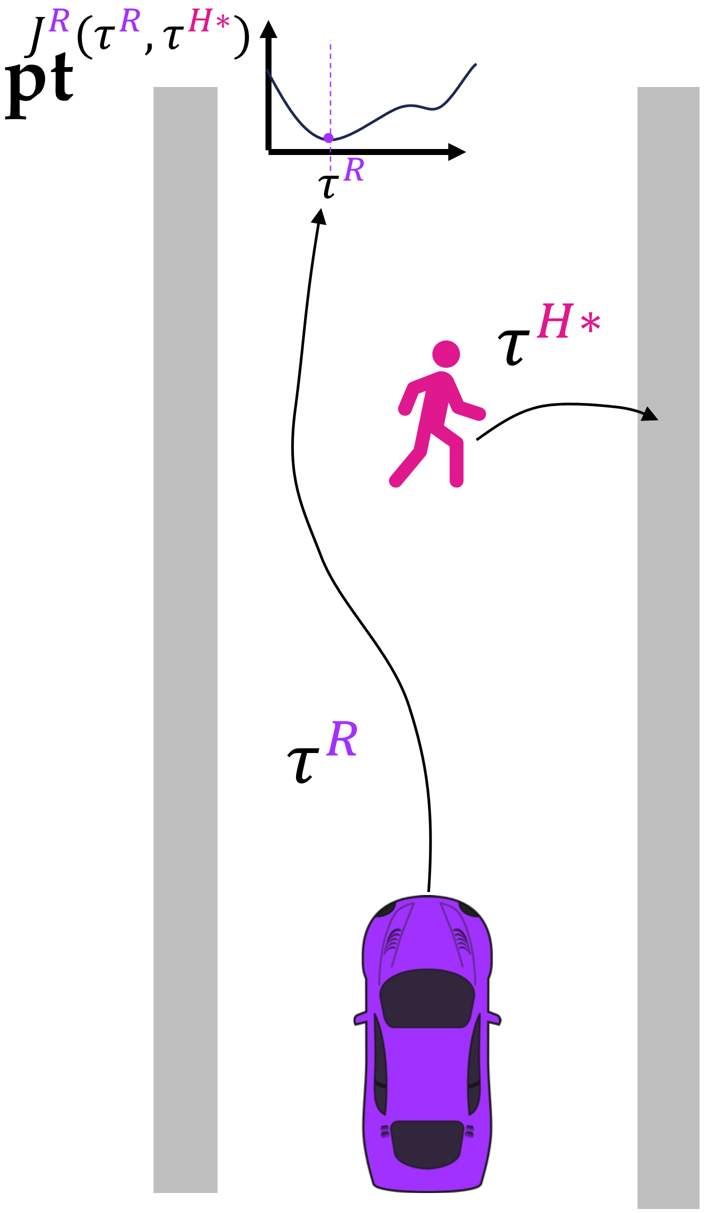

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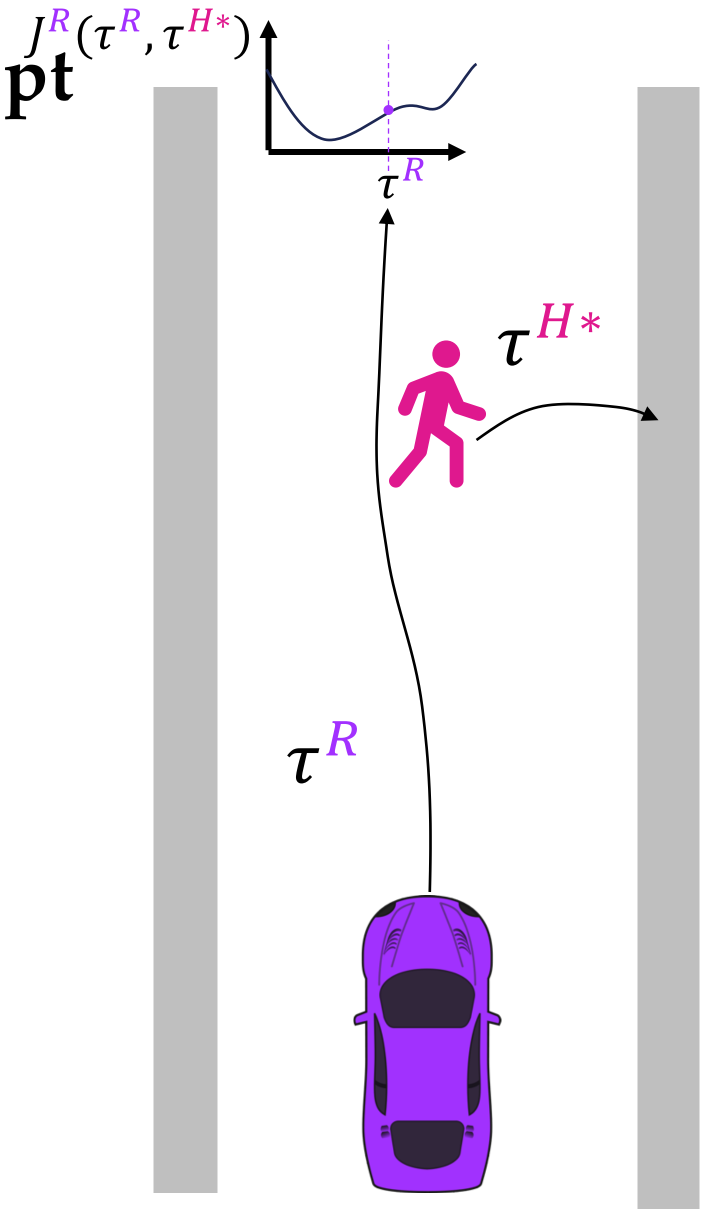

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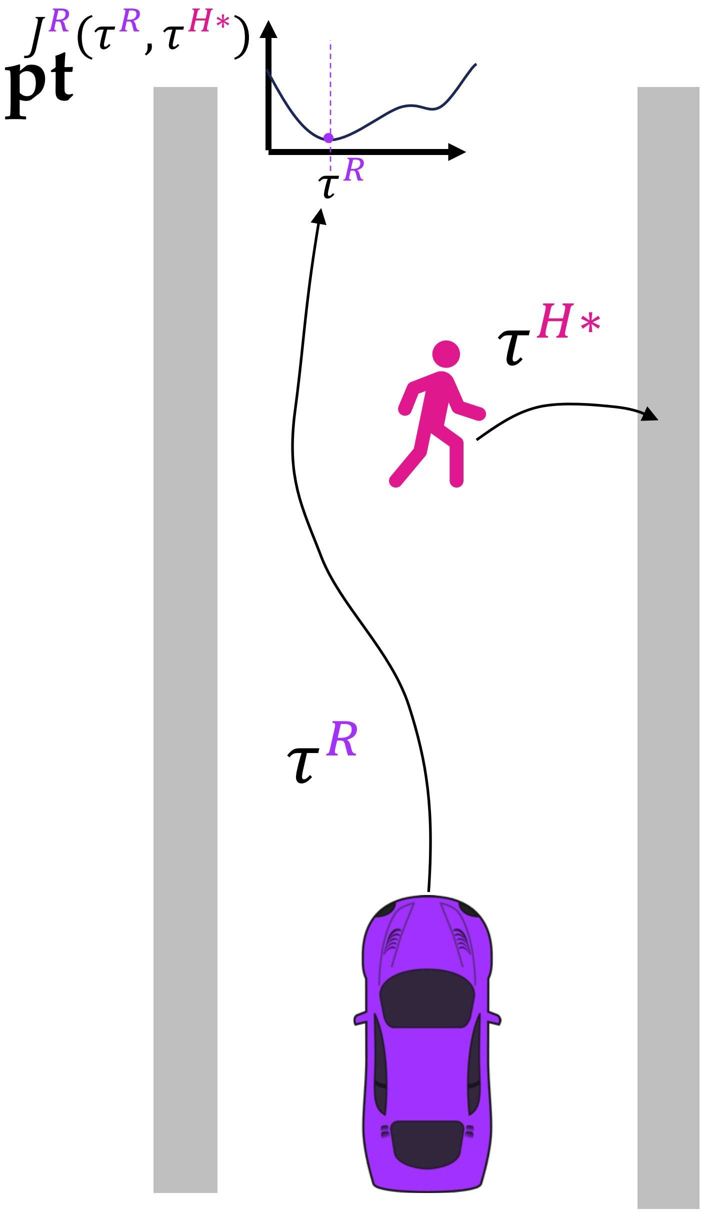

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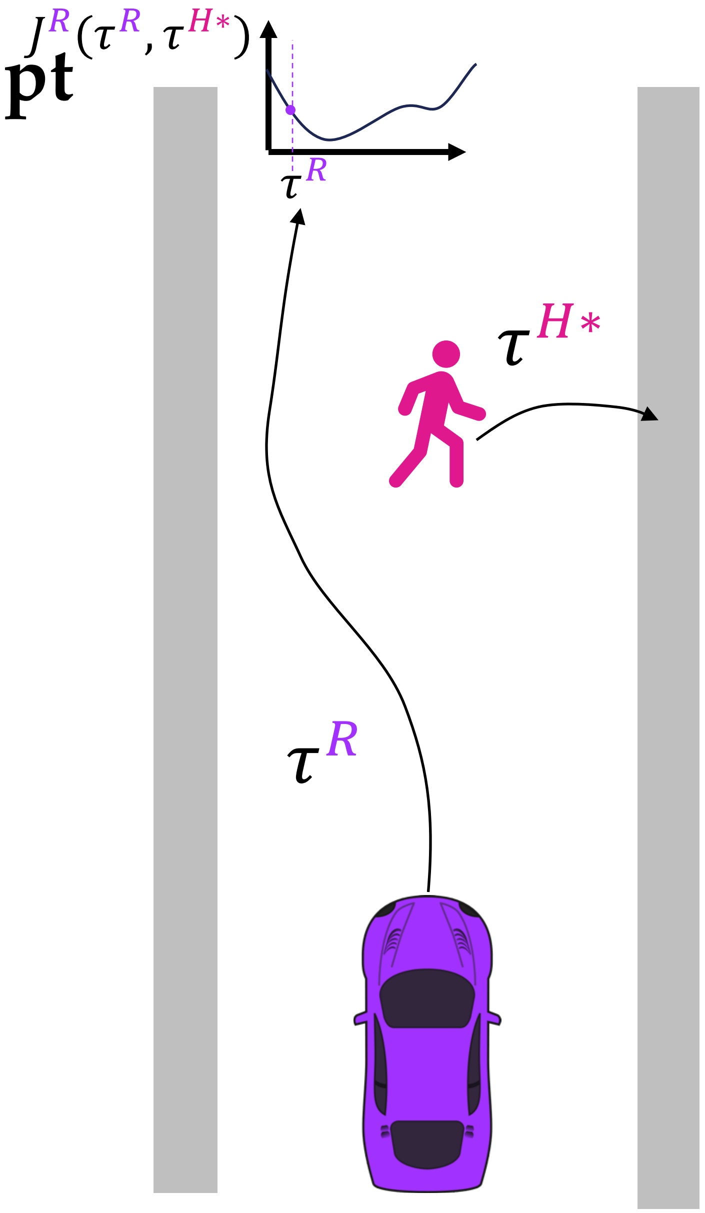

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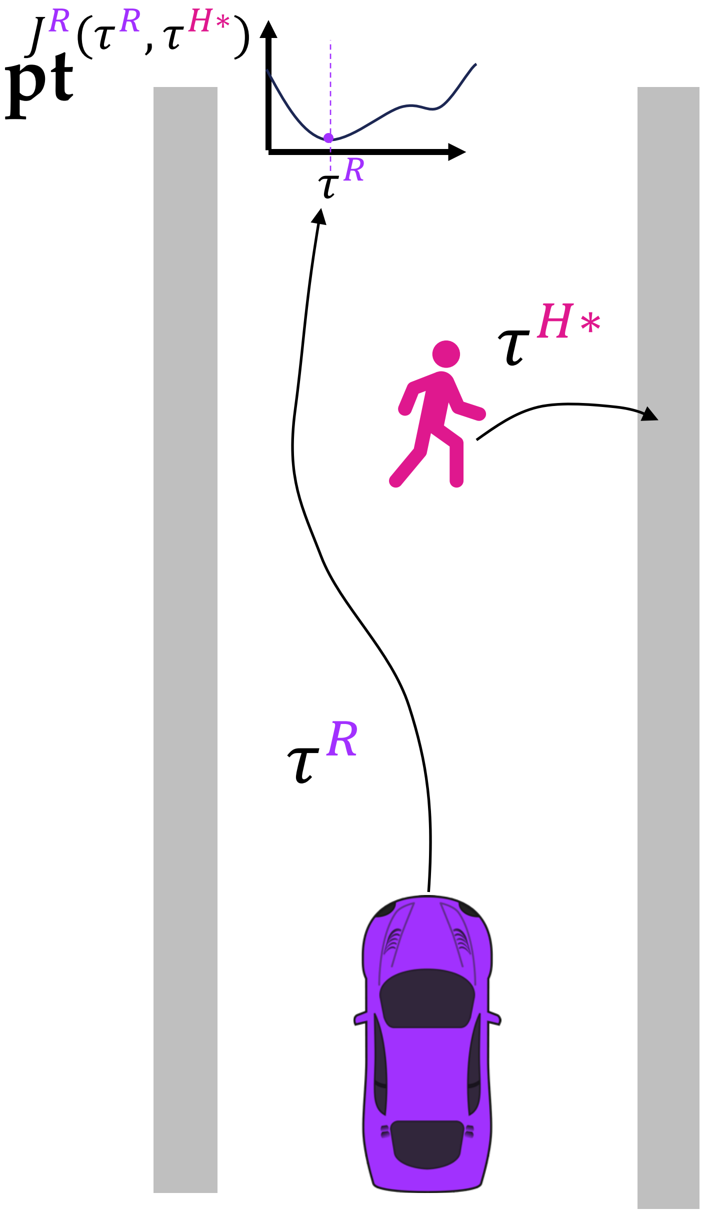

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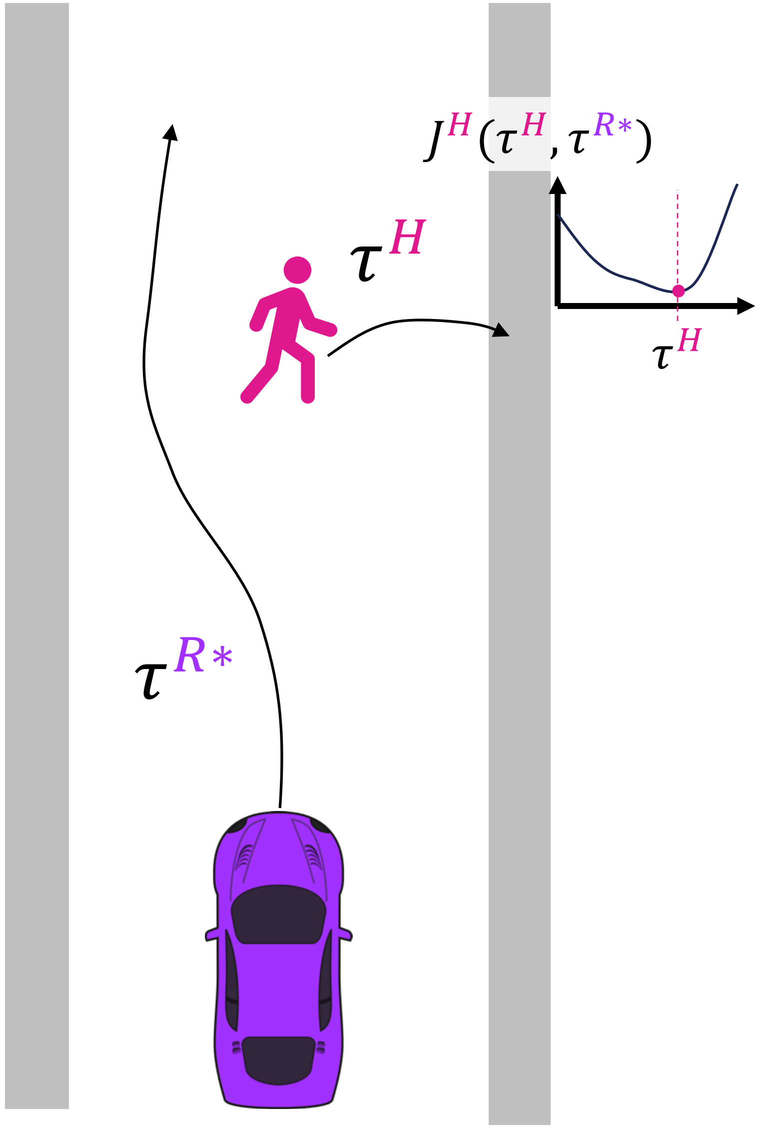

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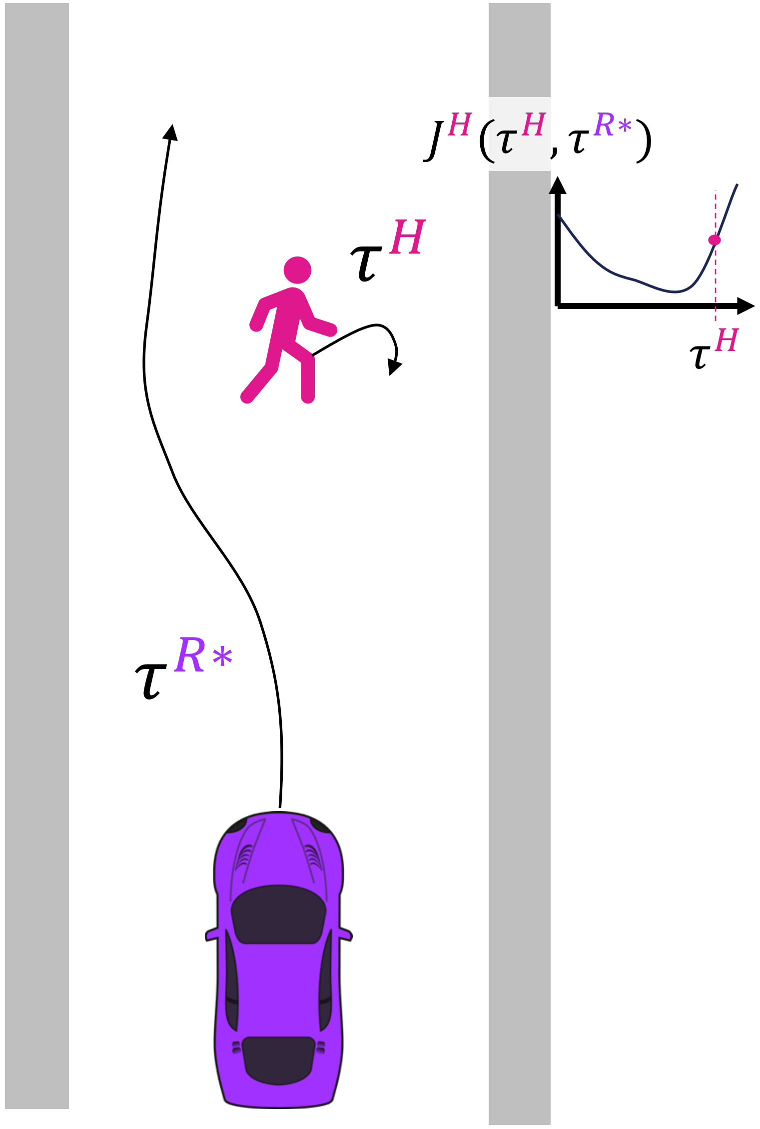

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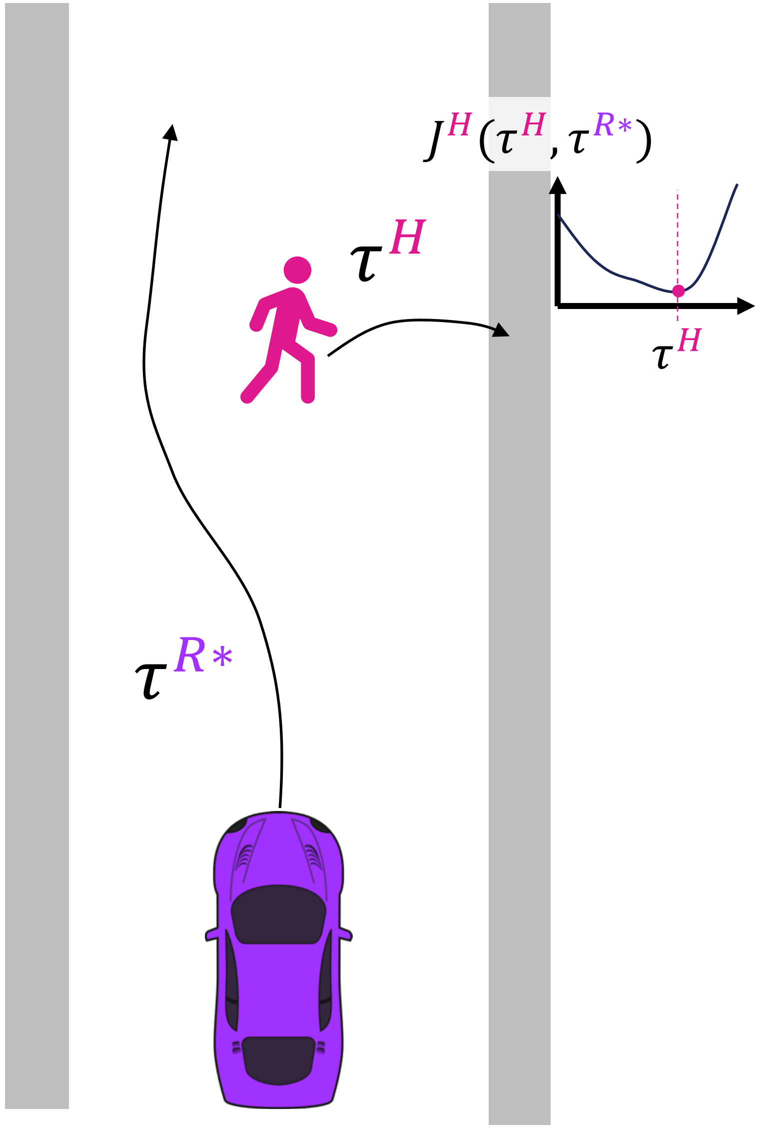

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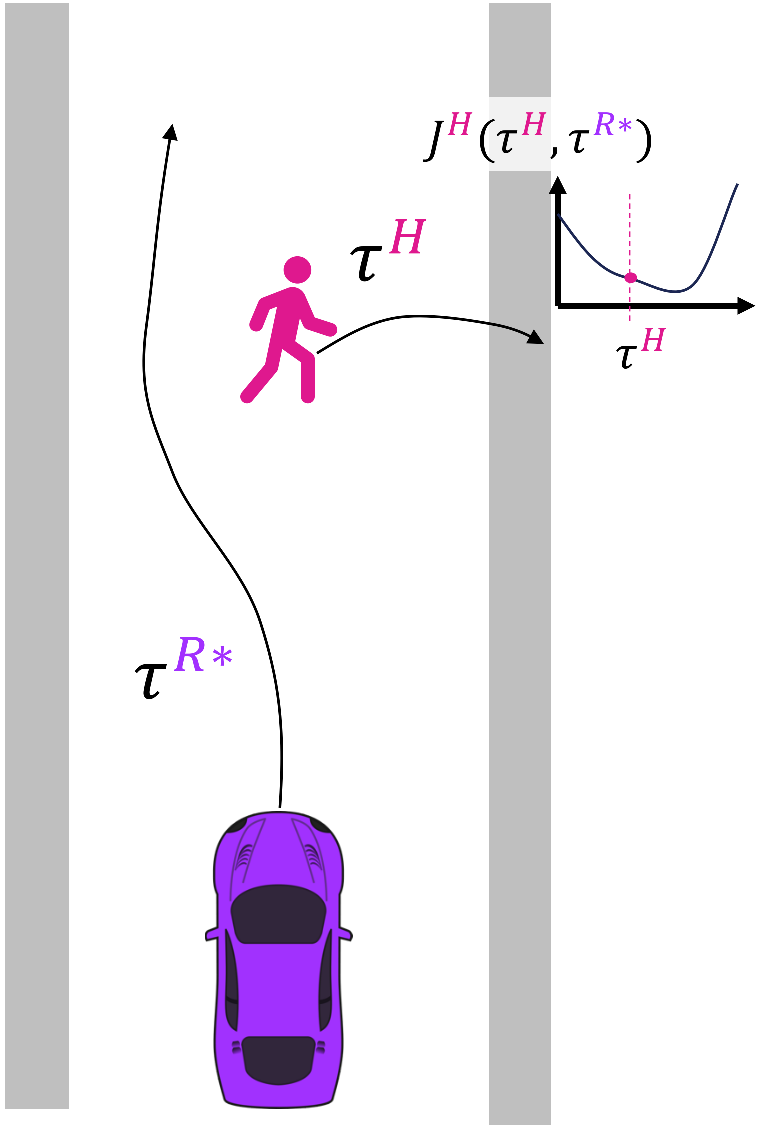

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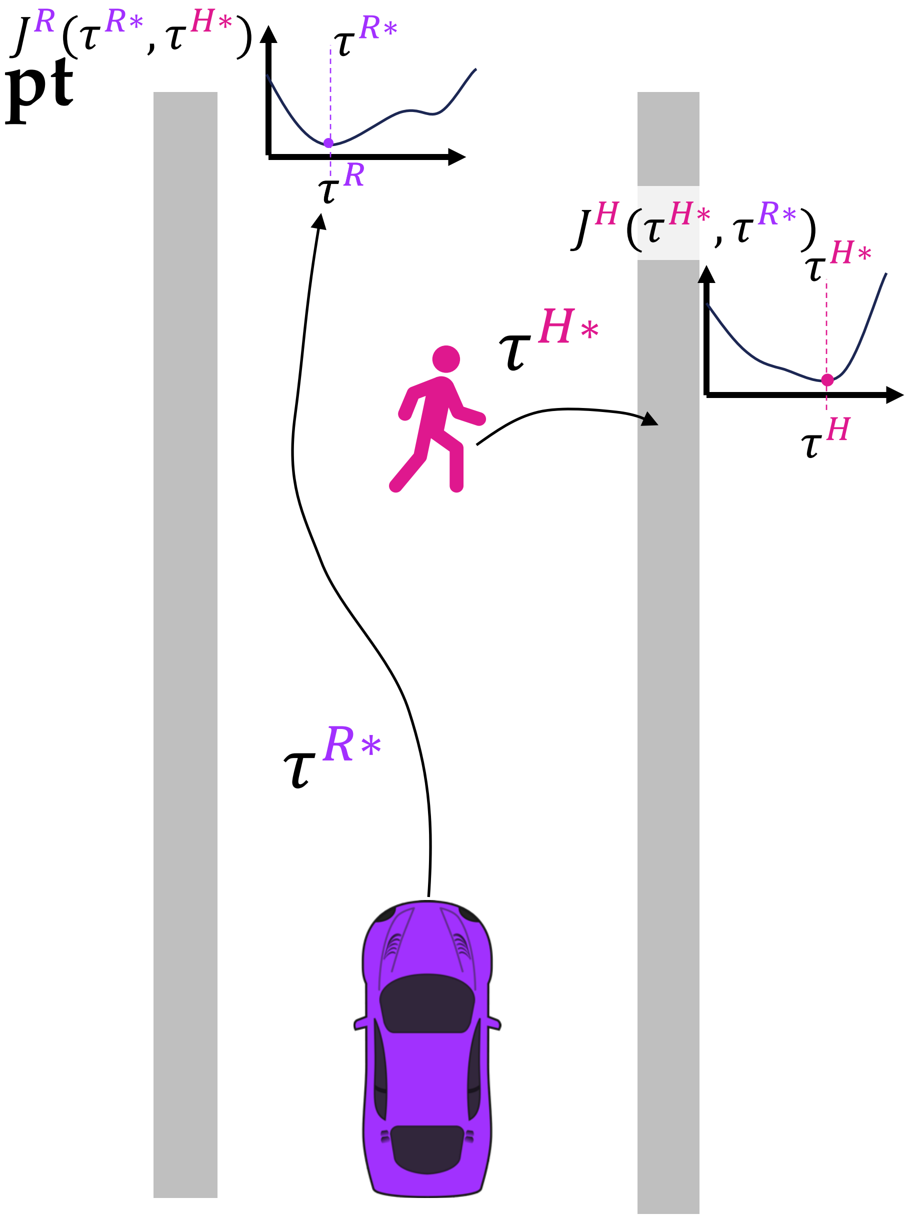

Model-Predictive Game-Play (MPGP)

The ***robot*** solves a game "in their head" ***at every time step***

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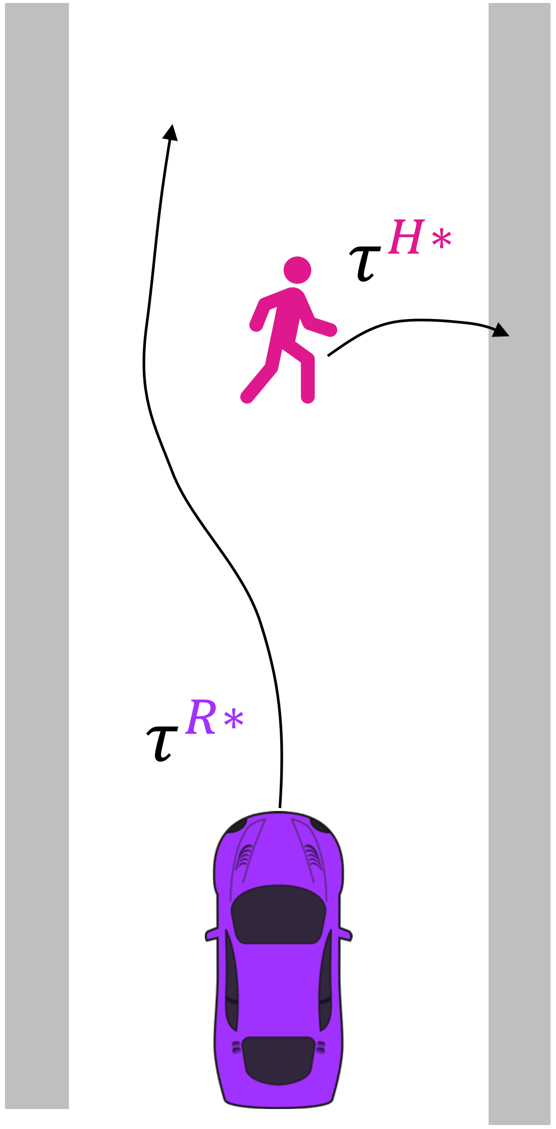

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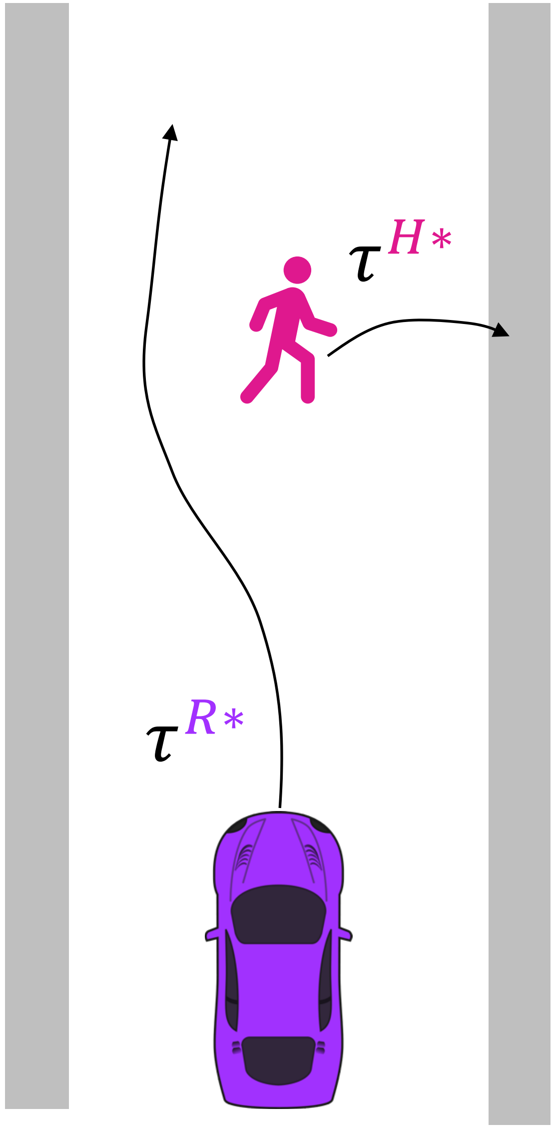

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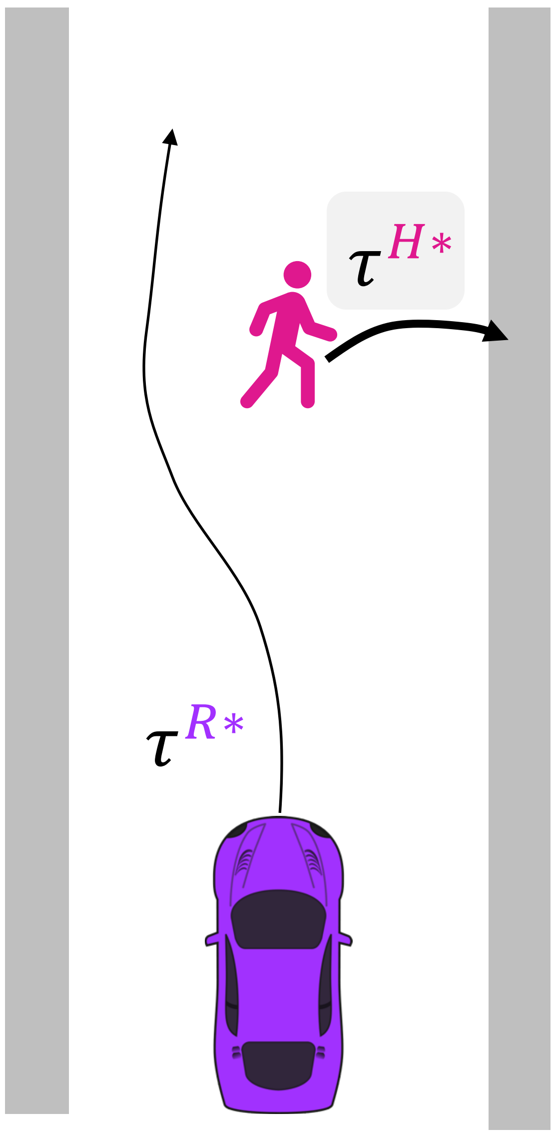

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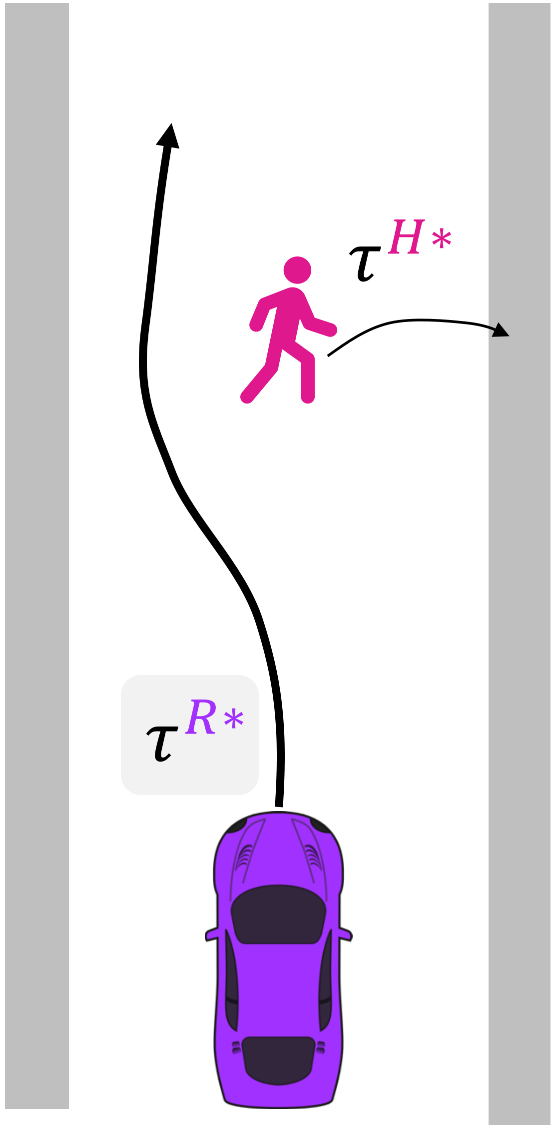

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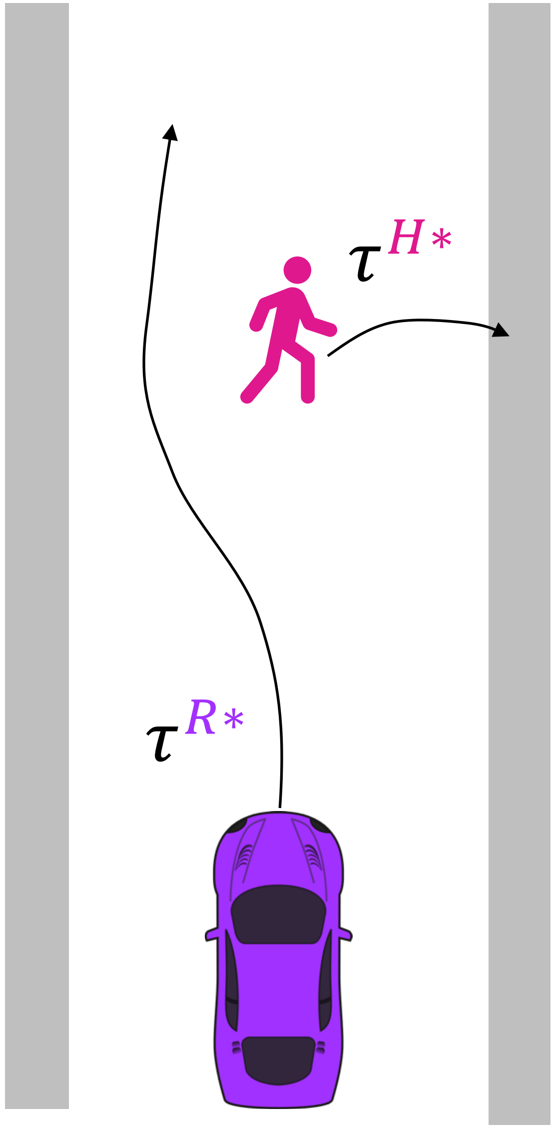
plan ← *computed jointly!* → prediction



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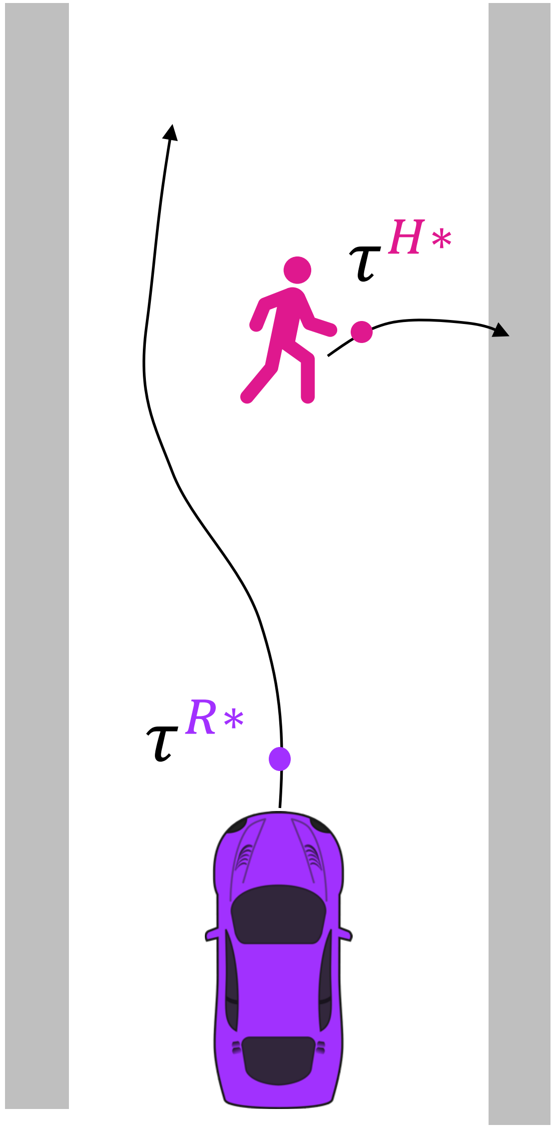

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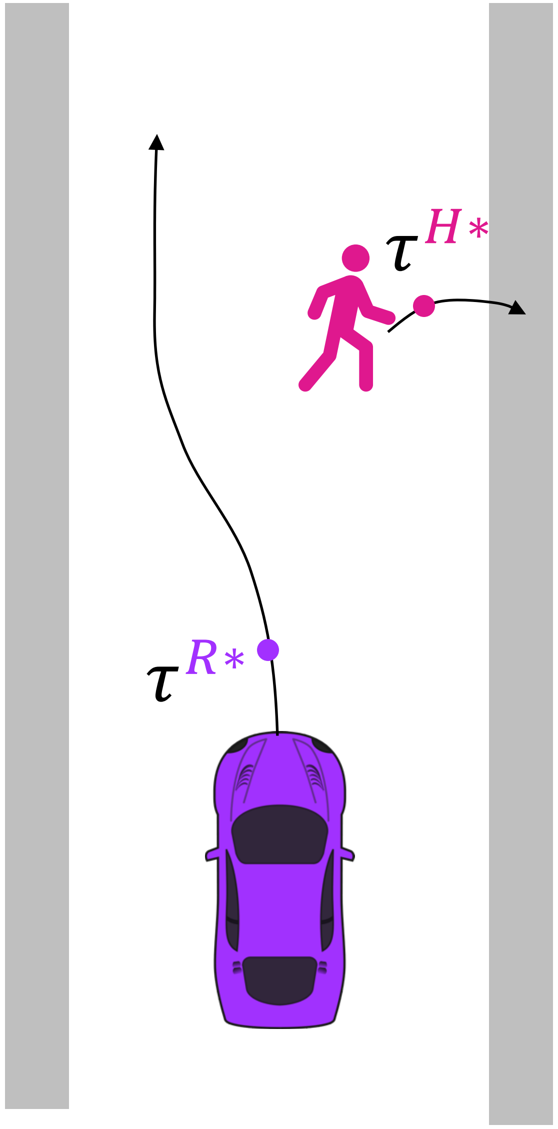

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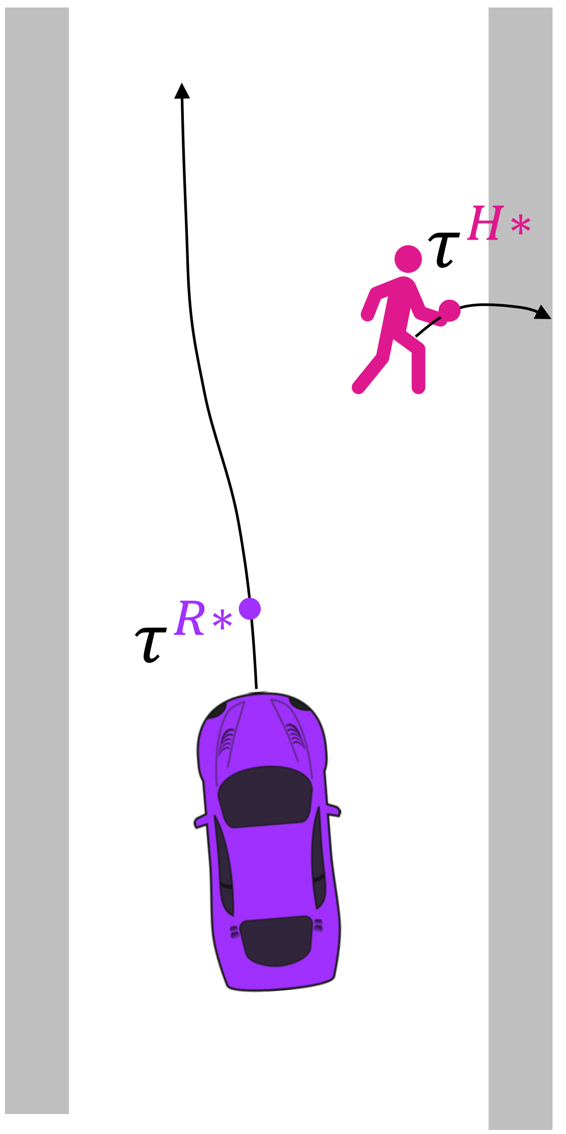

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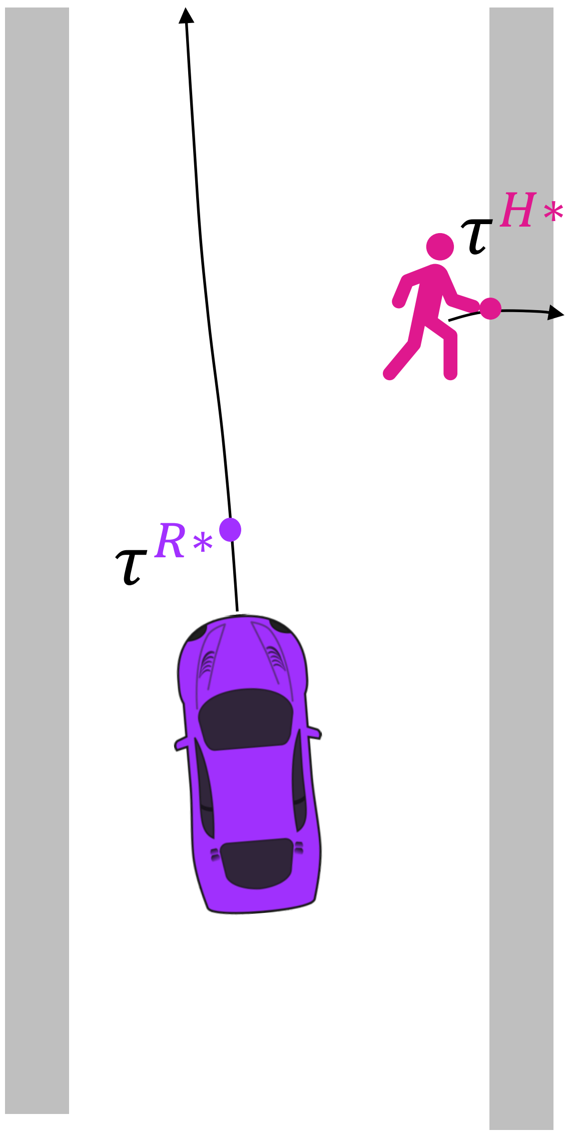


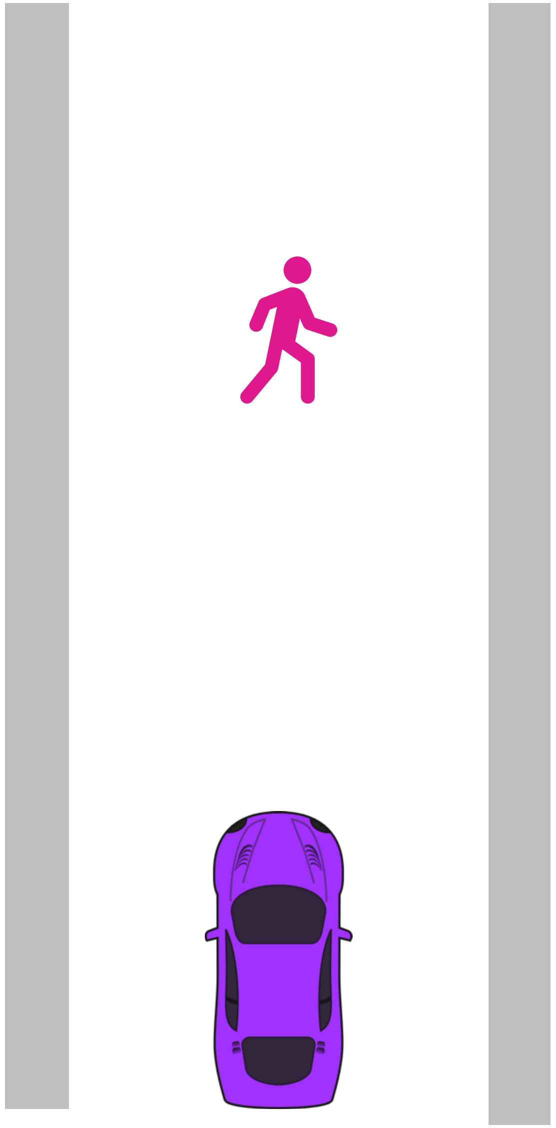
Solving *Open-Loop* Trajectory Games

Naïve Approach: Iterated Best Response

Generalized Nash Equilibrium conditions:

$$\tau^{i*} \in \mathcal{S}^i\left(\tau^{\neg i*}\right),\ i \in \{H, R\}$$

Challenge: τ^{i*} depends on $\tau^{\neg i*}$ and vice-versa!



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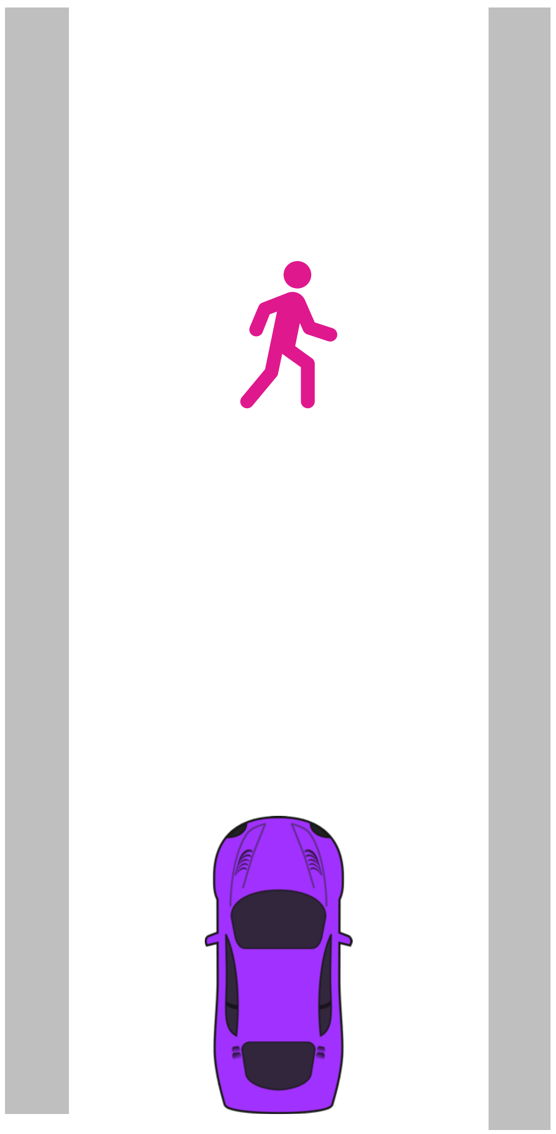
Key Idea: Start with an initial guess $\boldsymbol{\tau}_1 = \hat{\boldsymbol{\tau}} = (\hat{\tau}^1, \hat{\tau}^2, \ldots, \hat{\tau}^N)$;
exercise the equilibrium conditions as an update rule!



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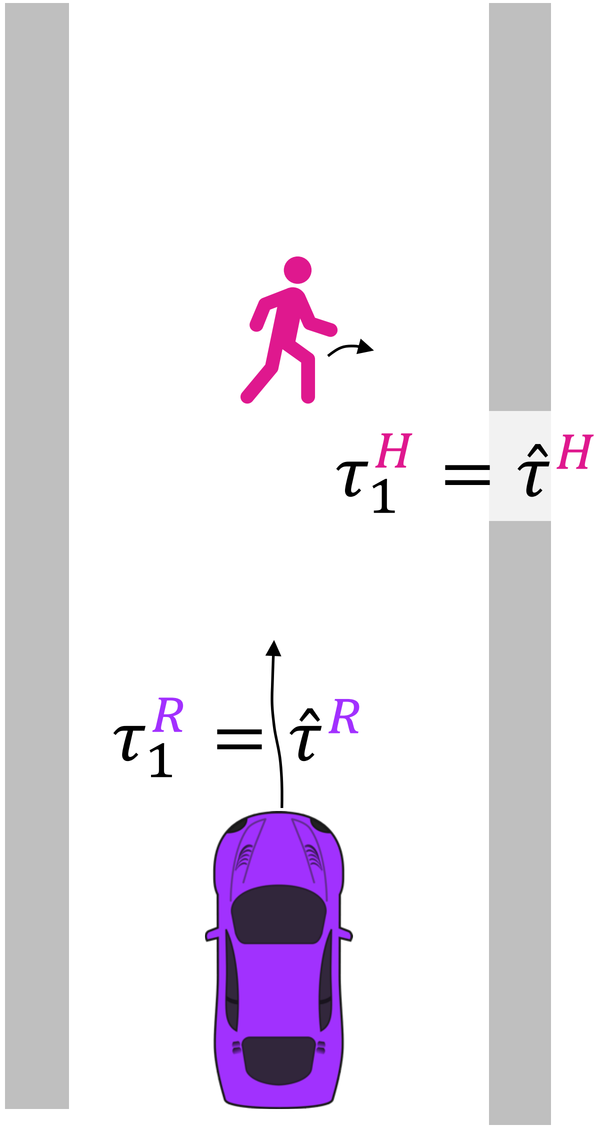
$$\tau_1^H = \hat{\tau}^H \qquad \tau_1^R = \hat{\tau}^R$$



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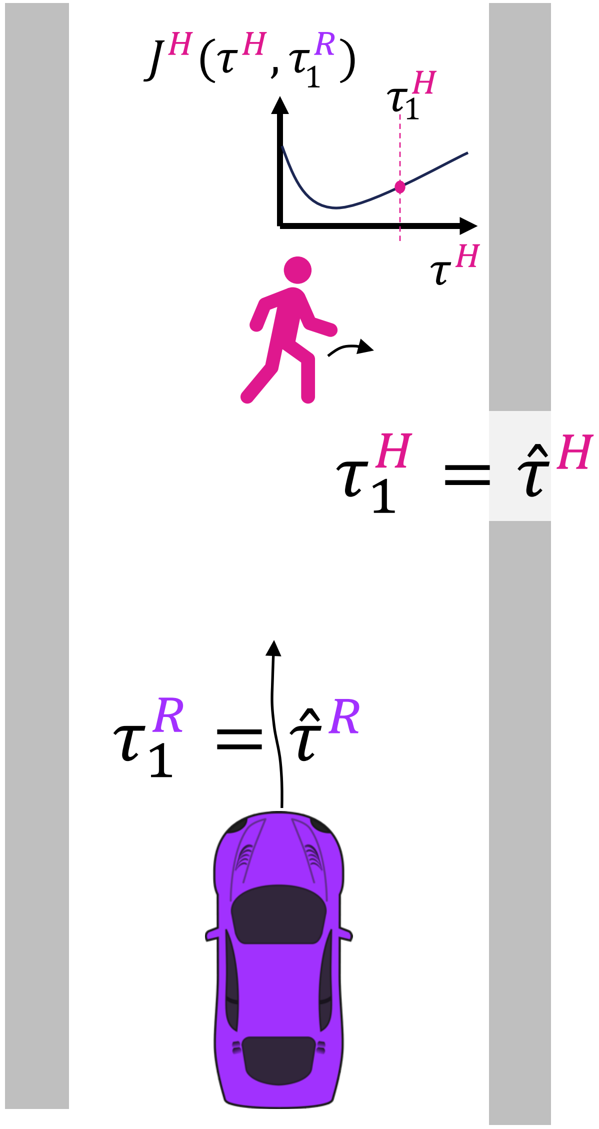

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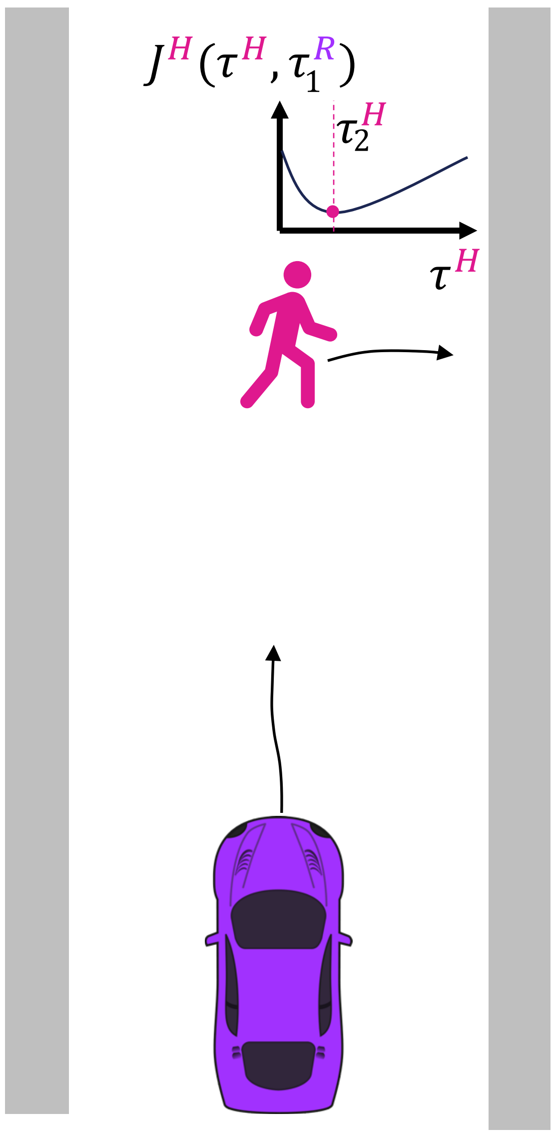

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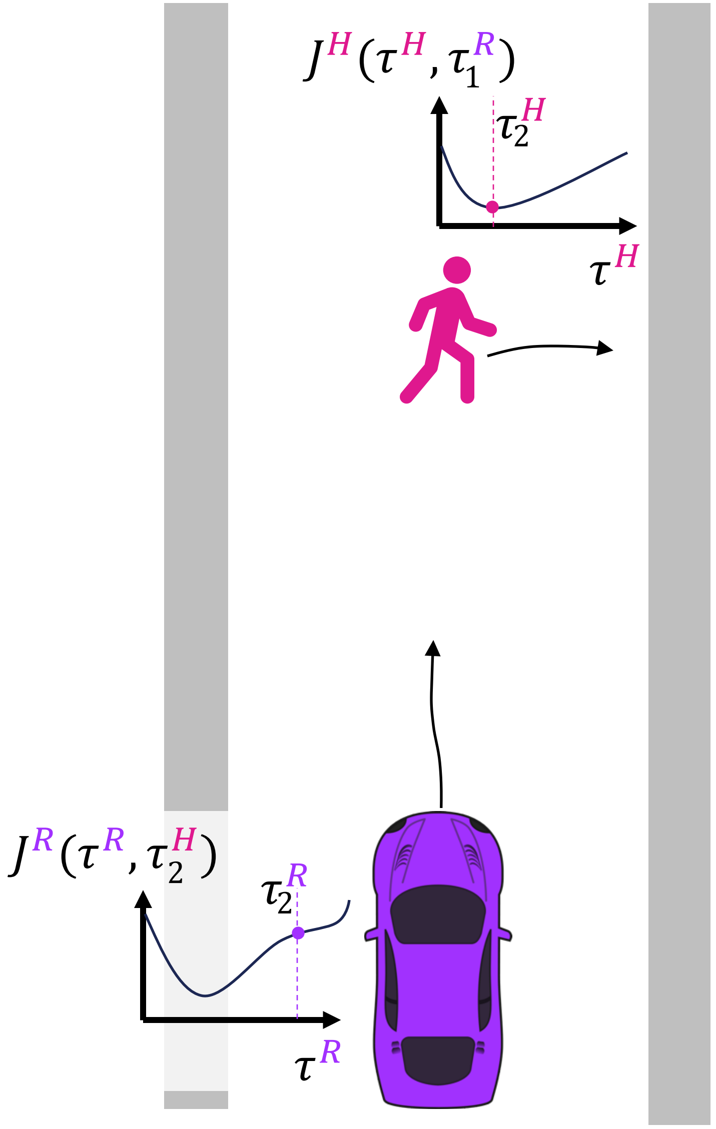
$$\tau_3^H = \tau_2^H$$



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exercise the equilibrium conditions as an update rule!

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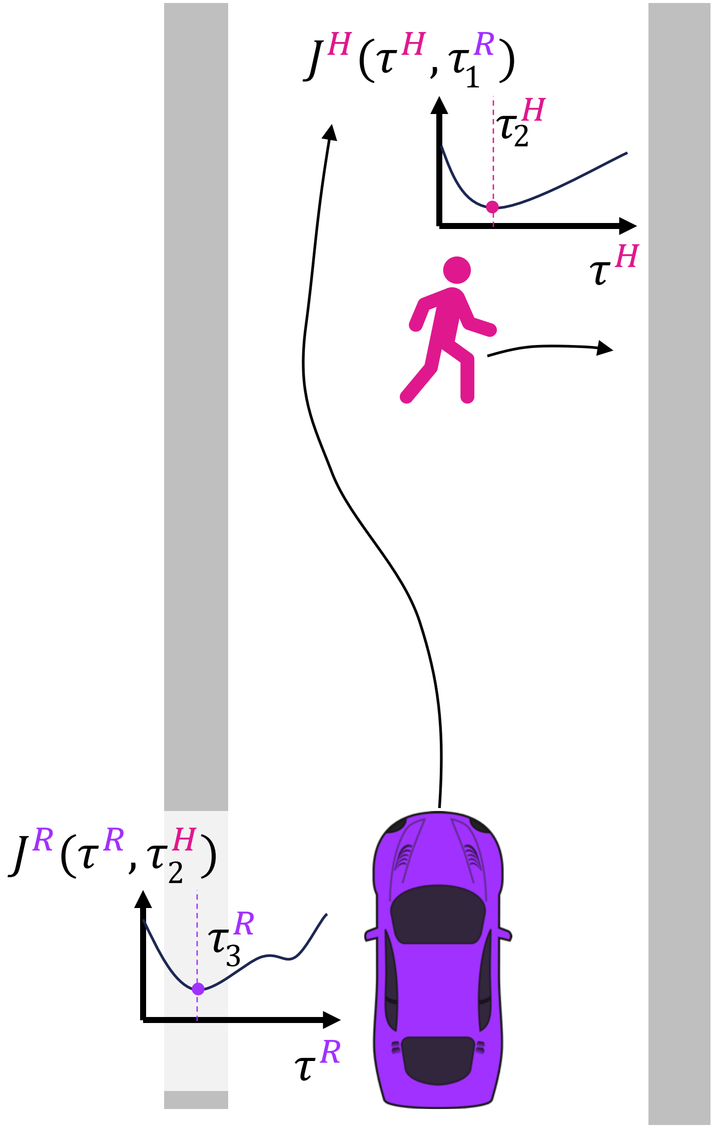

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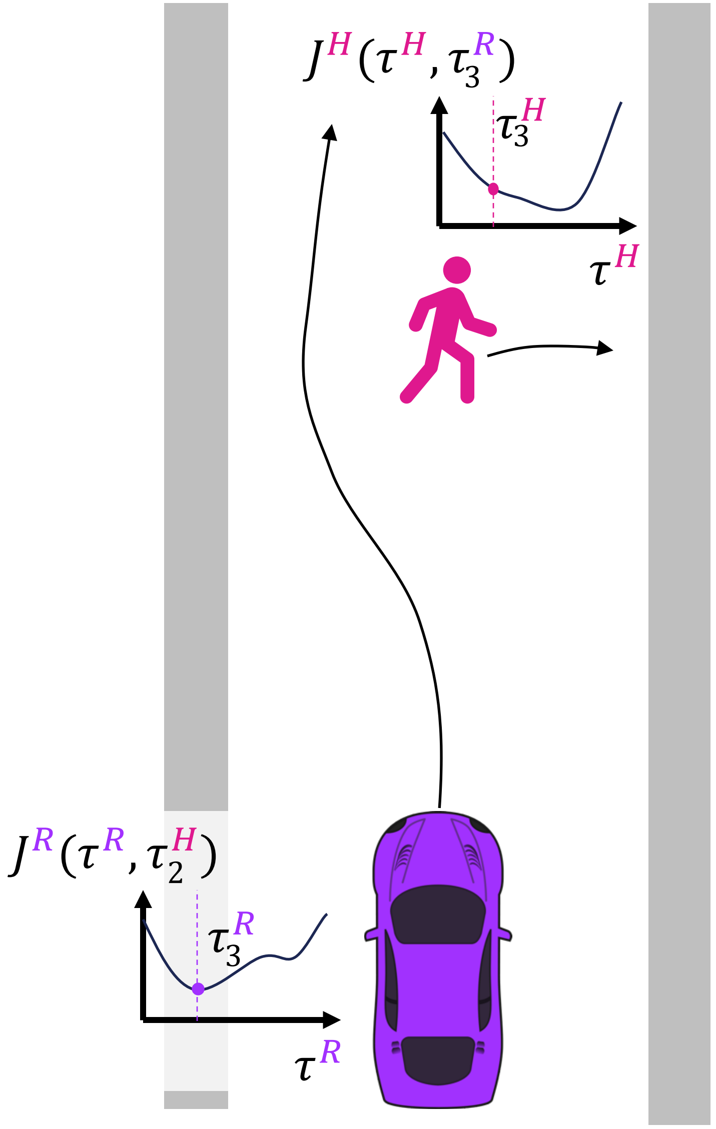

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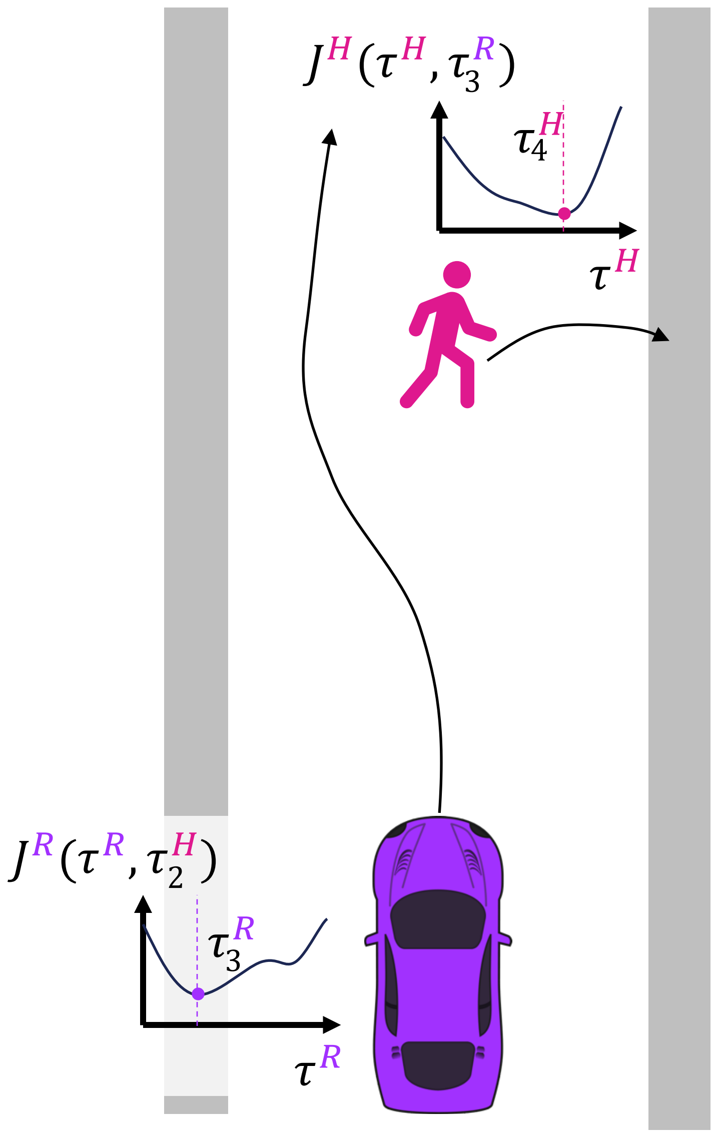

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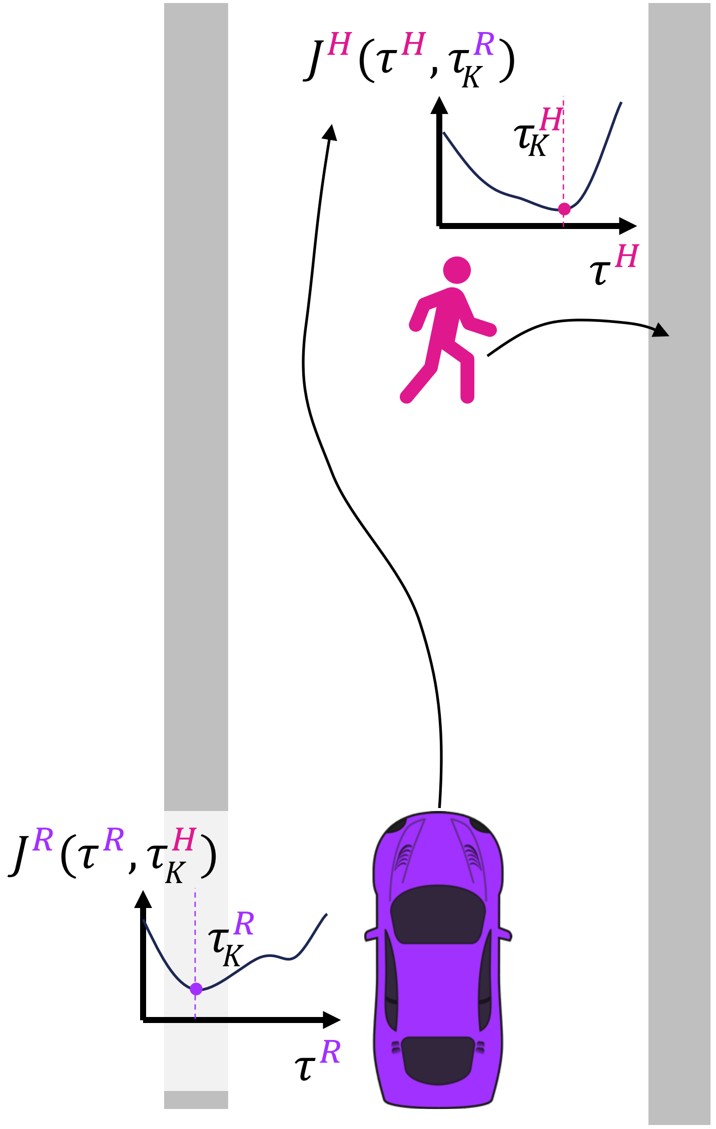

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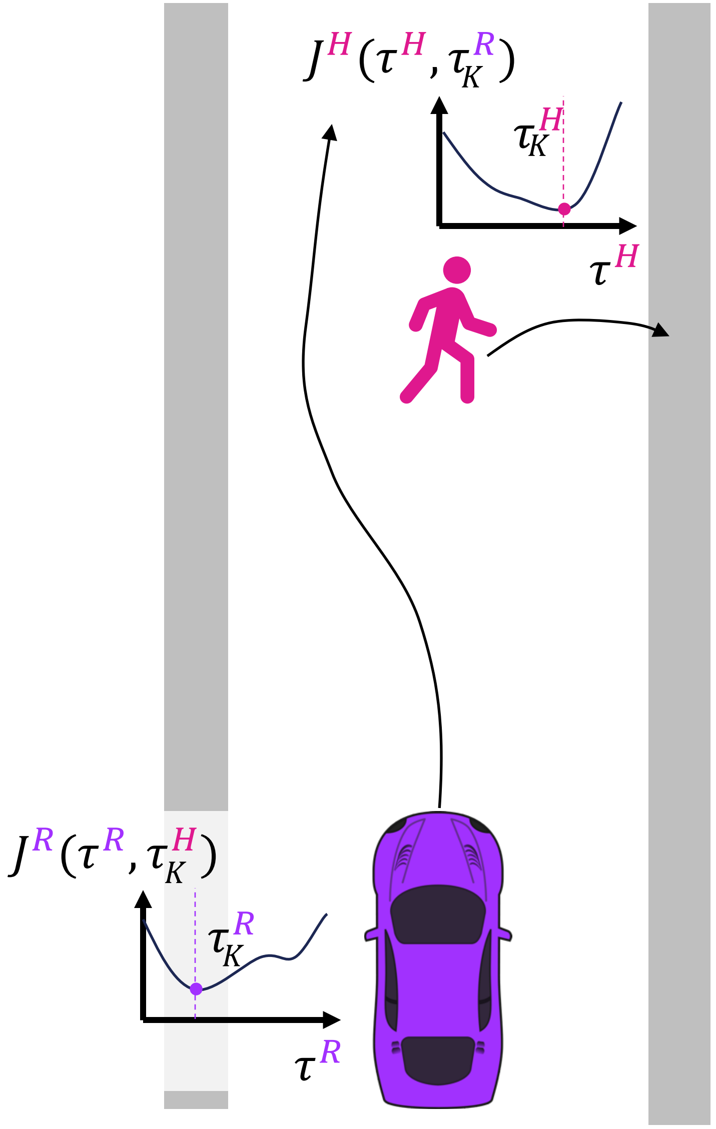
$\Downarrow$

Satisfies Nash conditions $\quad \tau_K^i = \tau^{i*} \in \mathcal{S}^i\left(\tau^{\neg i*}\right),\ i \in \{H, R\}$



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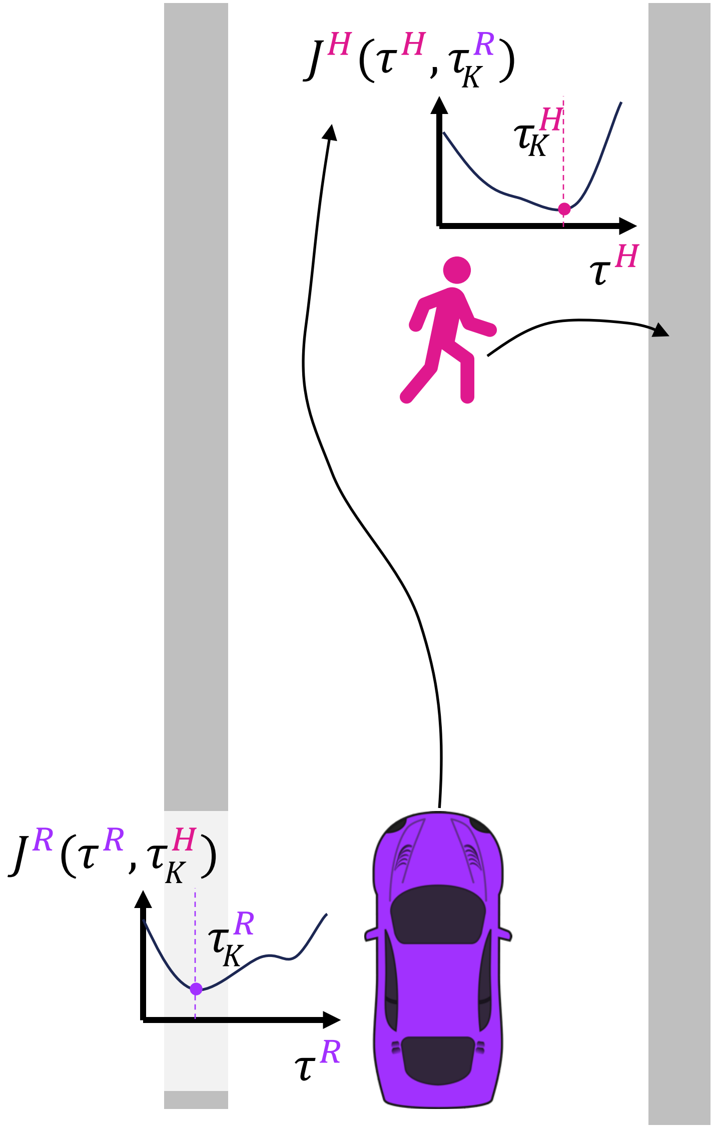

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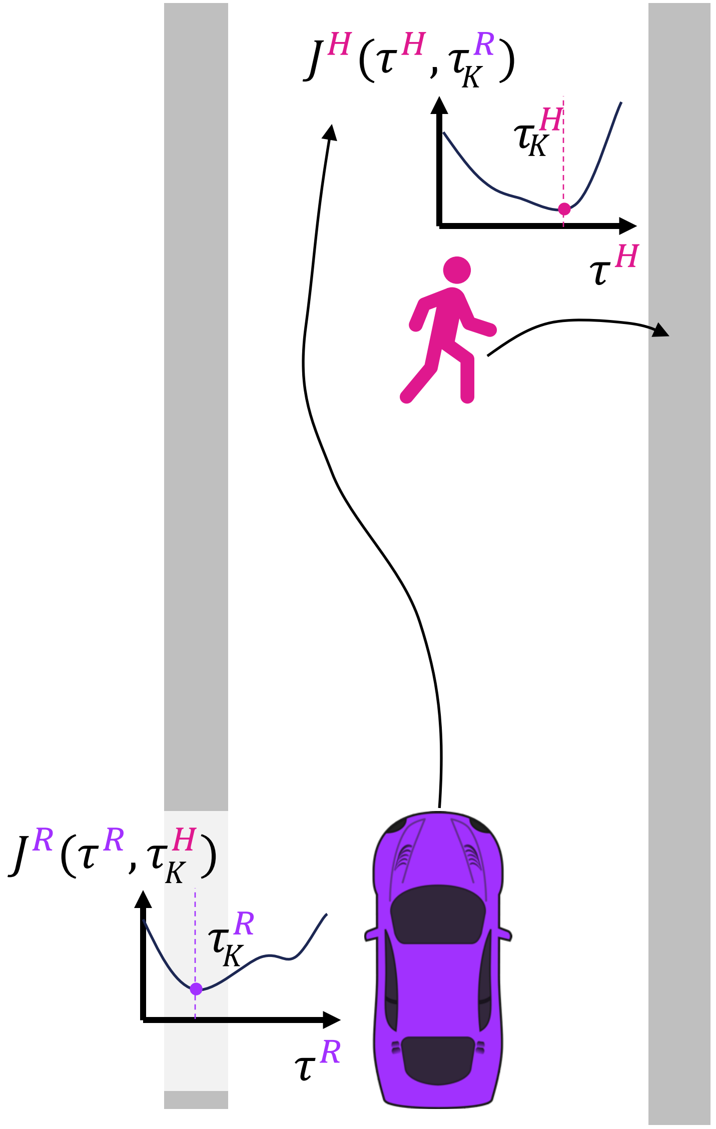

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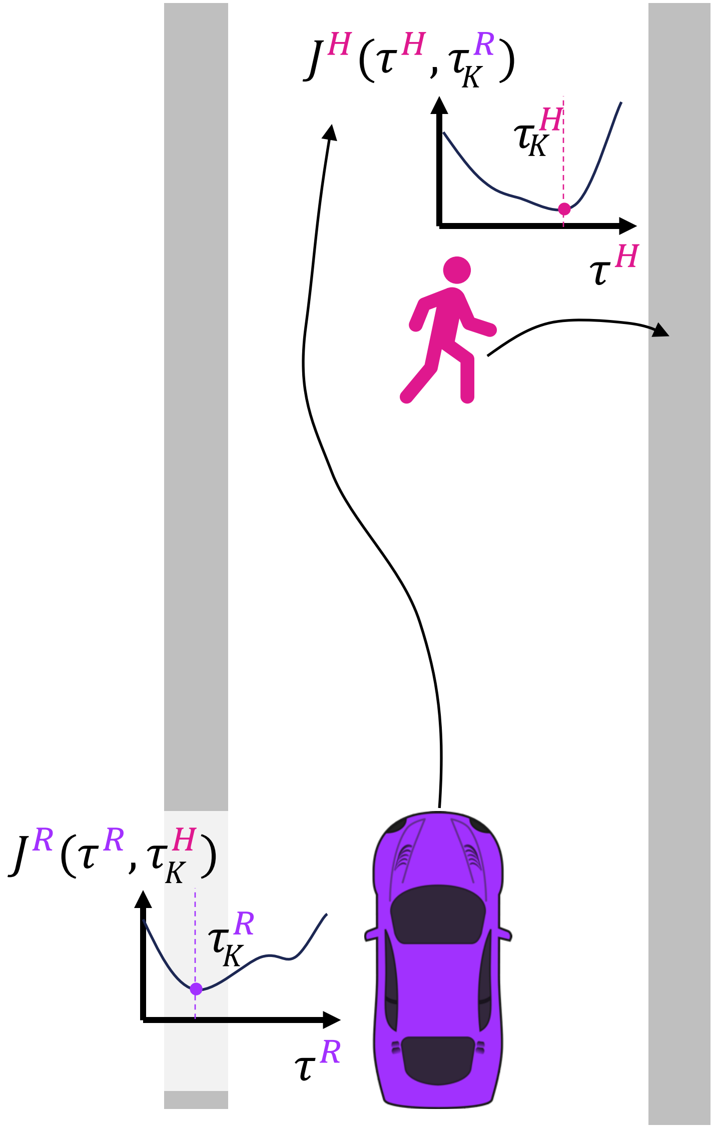

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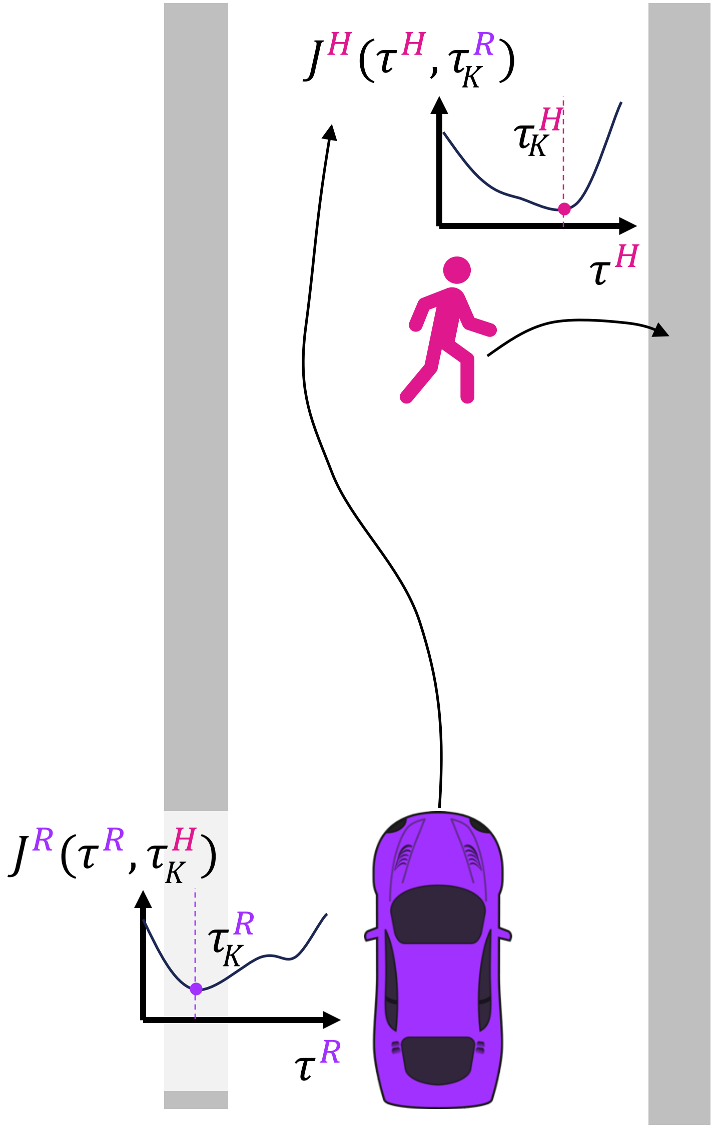
Can we do better?



Solving Open-Loop Games as Mixed Complementarity Problems

Key Idea: Search for *trajectory profile* that satisfies the *coupled KKT conditions*.

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Player i

Optimization Problem:

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\begin{aligned}
\min_{\tau^i} \quad & J^i(\tau^i, \tau^{\neg i}) \\
\text{s.t.} \quad & \tau^i \in \mathcal{K}^i(\tau^{\neg i})
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Lagrangian:

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Coupled KKT system:

$$\forall i \in [N] = \begin{cases} \nabla_{\tau^i} \mathcal{L}^i = 0, \\ 0 \leq h^i \perp \lambda^i \geq 0. \end{cases}$$

stacked for all players

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General form:

$$\text{Given } F: \mathbb{R}^d \mapsto \mathbb{R}^d; \ell, u \in \mathbb{R}^d; \text{find } z \in \mathbb{R}^d \text{ s.t.}$$
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$$\Leftrightarrow$$

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Coupled KKT system as MCP:

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z = \left[\begin{Bmatrix} \tau^i \\ \lambda^i \end{Bmatrix} \forall i\right], \qquad F(z) = \left[\begin{Bmatrix} \nabla_{\tau^i}\mathcal{L}^i \\ h^i \end{Bmatrix} \forall i\right], \qquad \ell = \left[\begin{Bmatrix} -\infty \\ 0 \end{Bmatrix} \forall i\right], \qquad \mathrm{u} = \left[\begin{Bmatrix} \infty \\ \infty \end{Bmatrix} \forall i\right].
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If ∇F is ***smooth and is sparse***, modern MCP solvers, e.g. PATH*, can find solutions rapidly!

Example: 5-player game, 25 time steps
3,208 decision variables, solution in **35 ms**

* [Dirkse 1995]

Beyond Open-Loop Information Structure:
Feedback Games

Why Care About Feedback?

Open-loop games

- Capture rich behavior, including collision avoidance etc.
- Receding-horizon takes care of prediction errors

But: Open-loop games cannot capture *"indirect interaction"*

Why Care About Feedback?

Open-loop games cannot capture *“indirect interaction”*

Example:

$$J^R(\tau^R, \tau^H) = \text{goalDistance}^R(\tau^R) + \text{collisionCost}(\tau^H, \tau^R)$$

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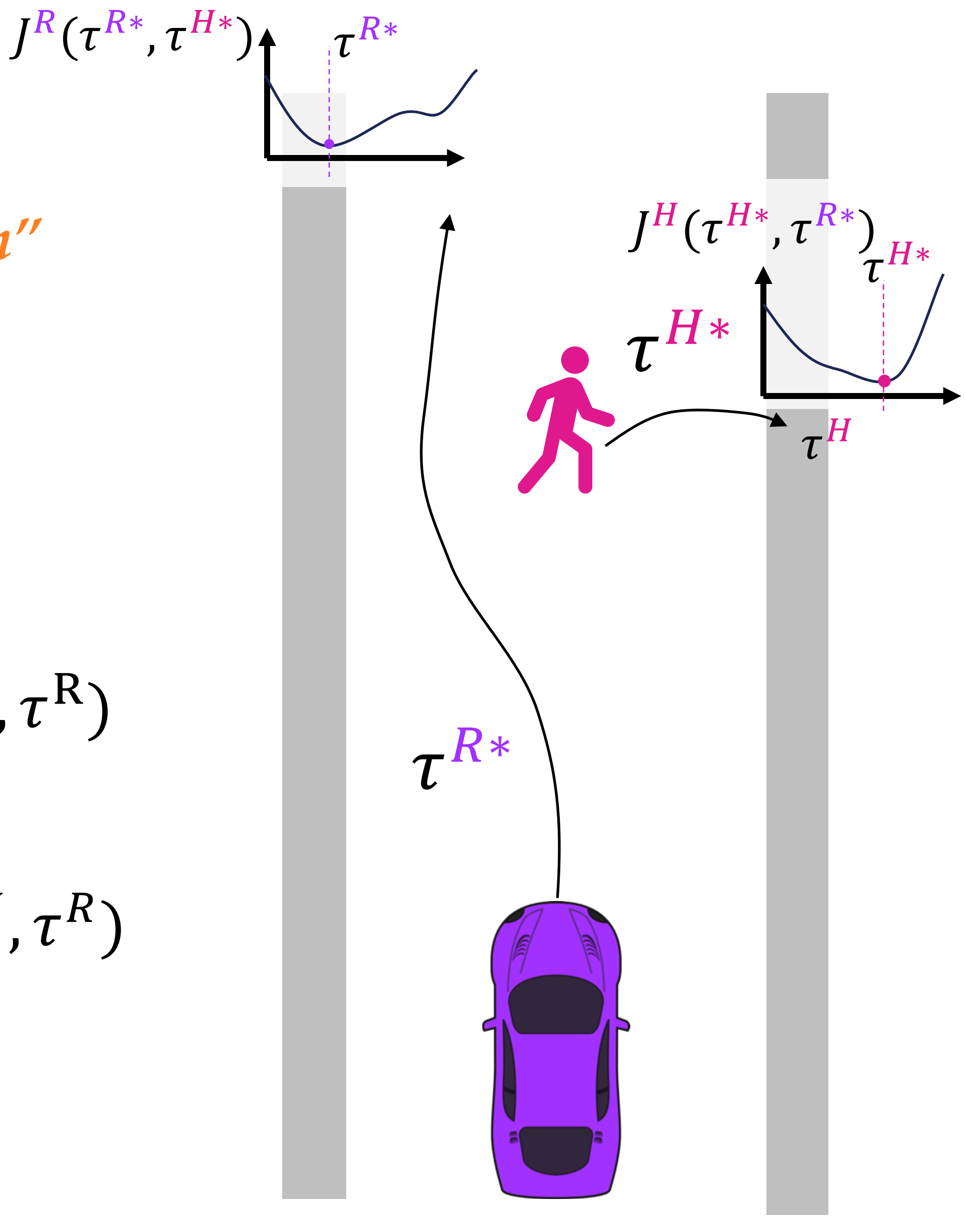

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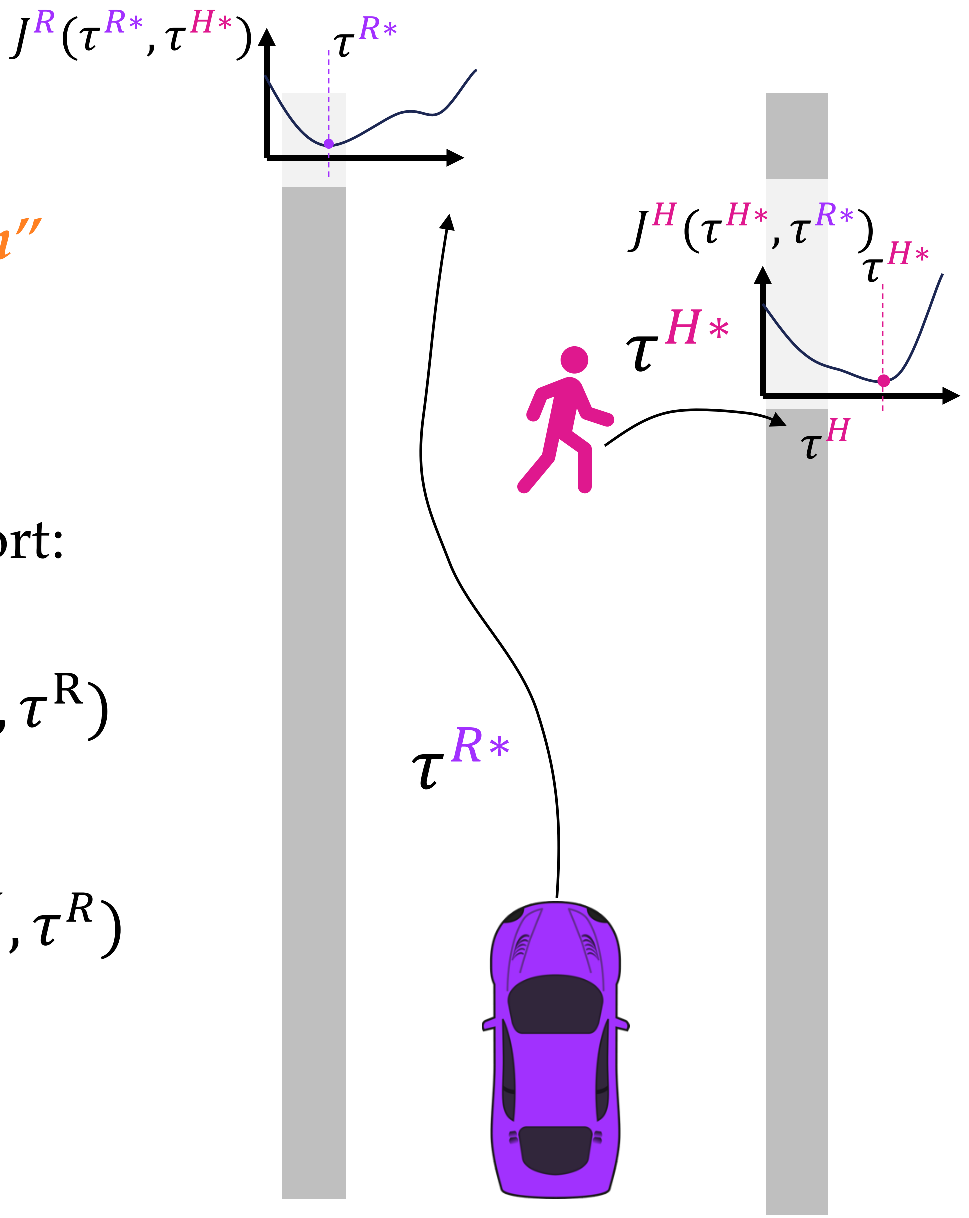

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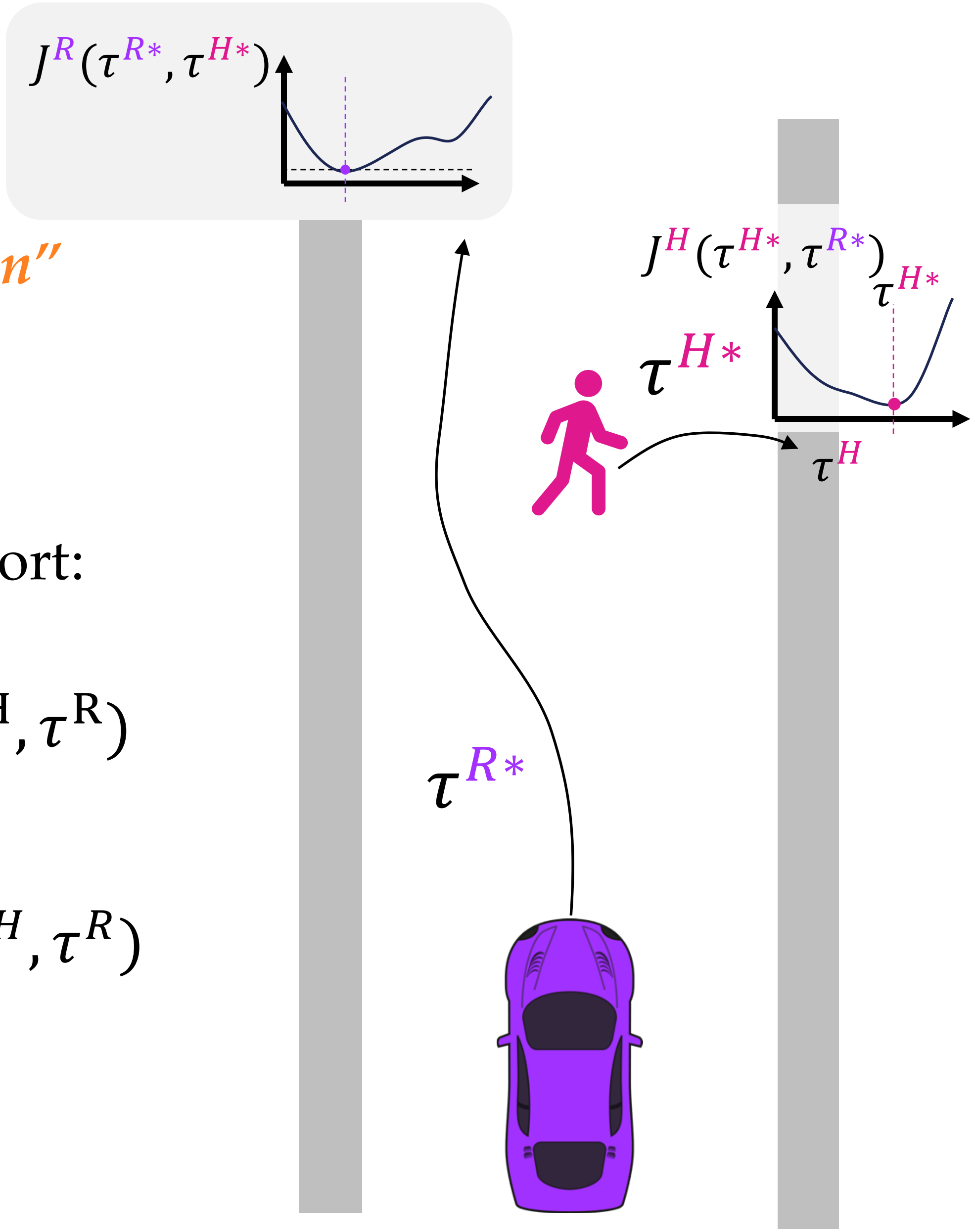

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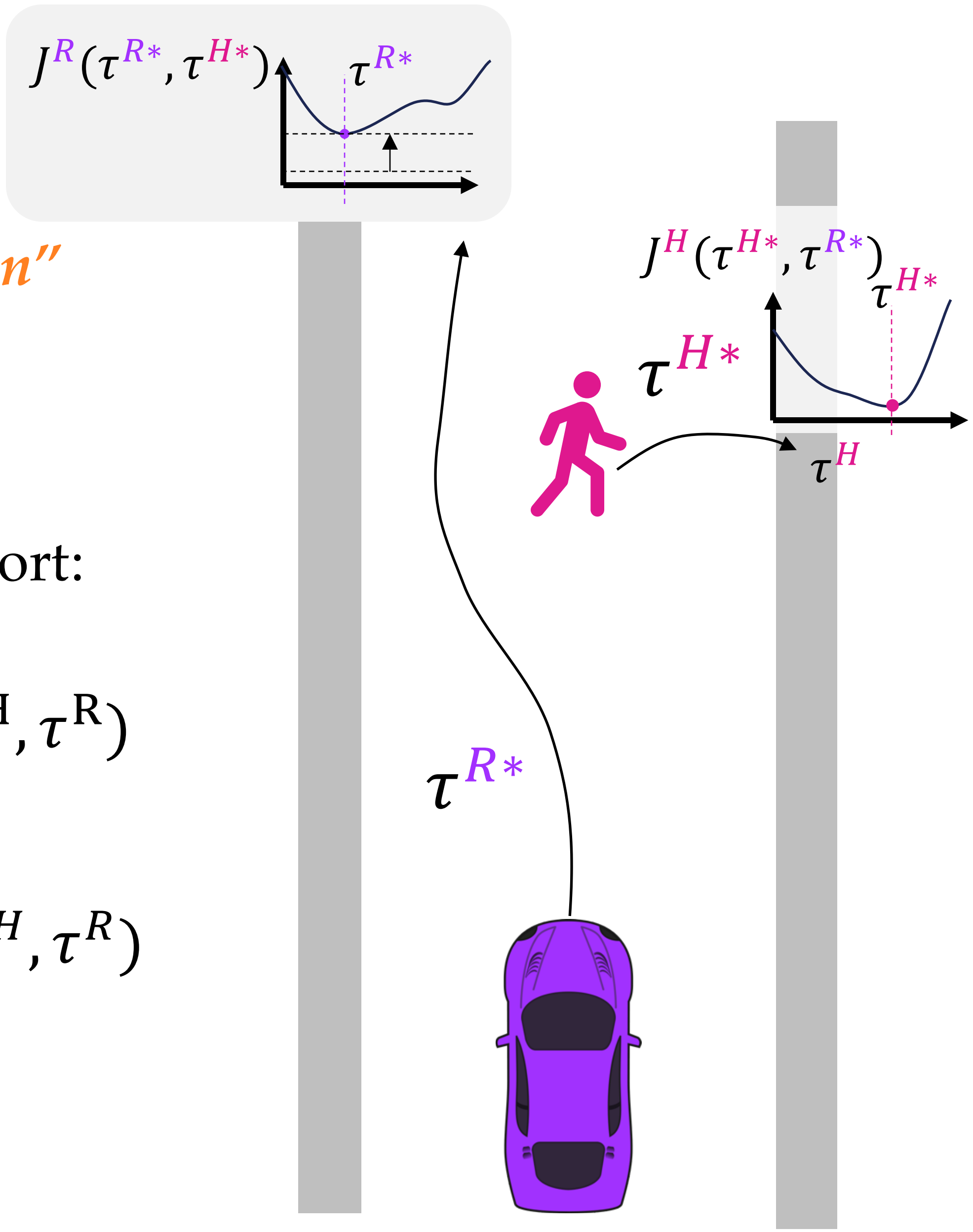
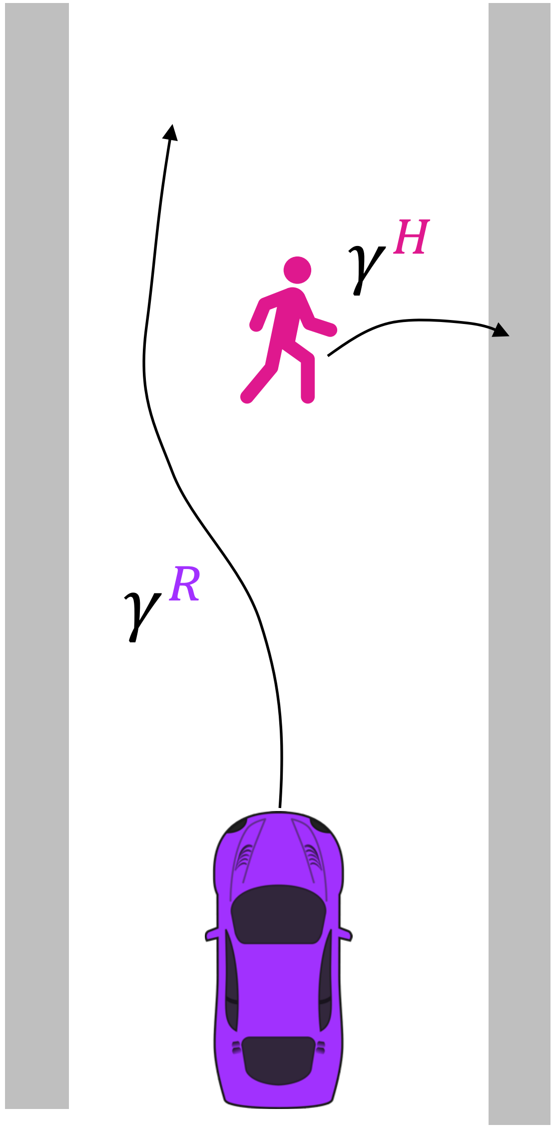


Feedback to the Rescue

Key ingredient: players reason about time-varying *feedback strategies:*

$$\Gamma^i \ni \gamma^i : \mathcal{X} \times [T] \rightarrow \mathcal{U}^i$$
$$u_t^i = \gamma^i(x_t, t)$$



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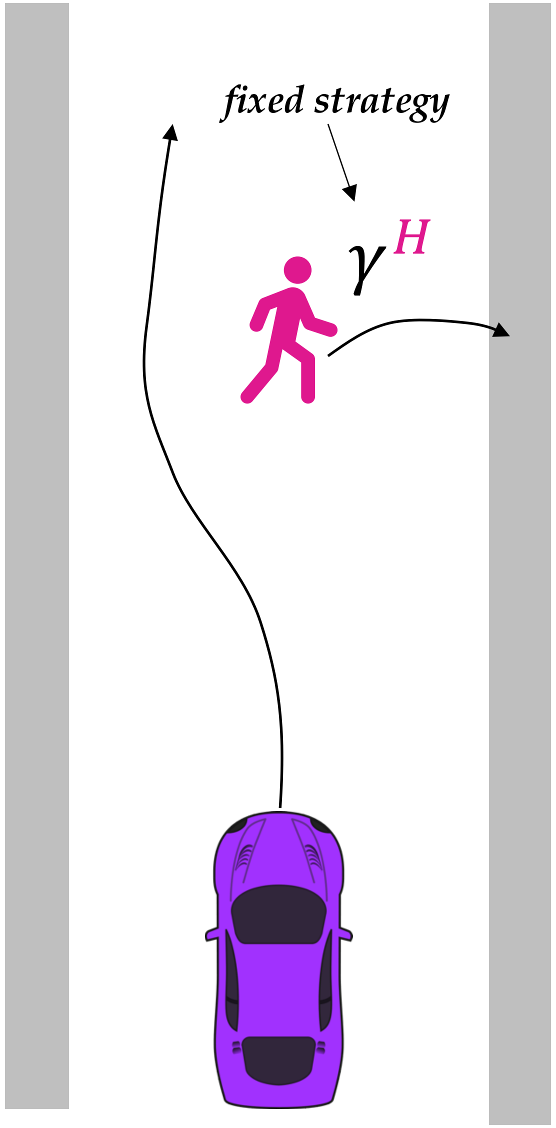

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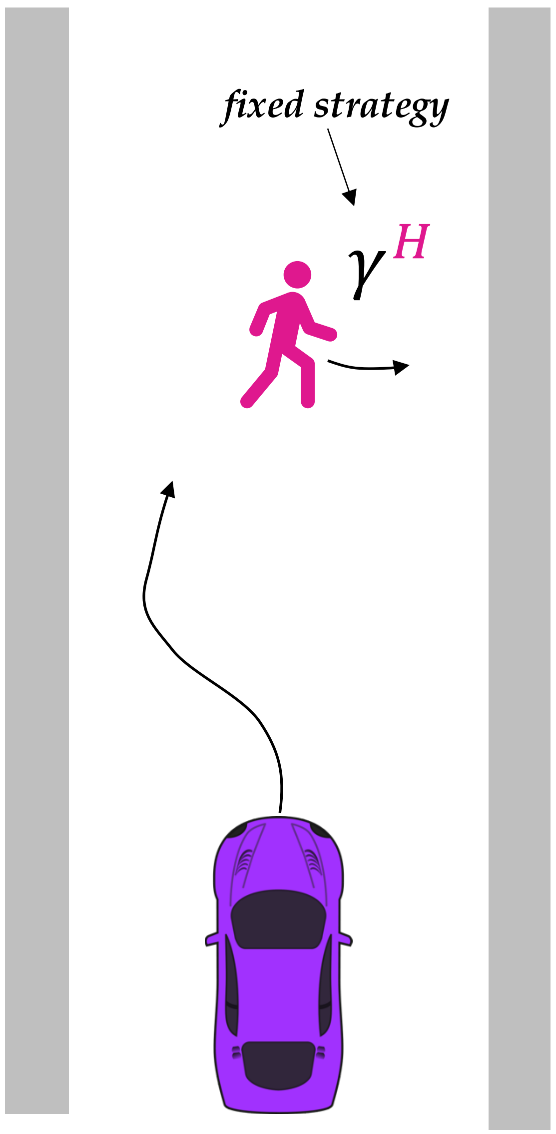

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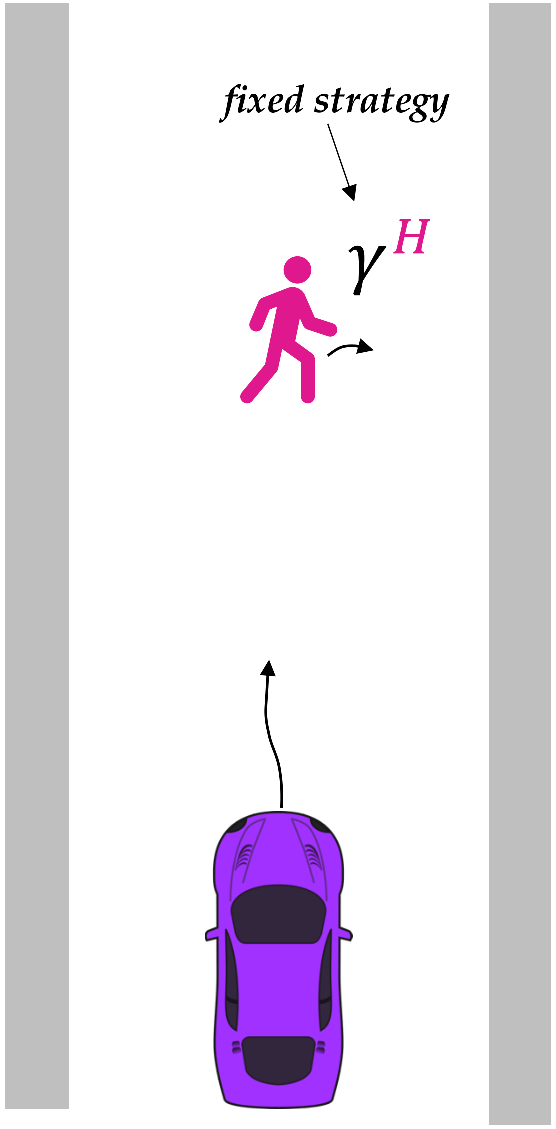

Generalized Feedback Nash Equilibria

Disclaimer: even a rigorous problem definition for feedback-GNE can be overwhelming.

TL;DR: **Feedback-GNE result in *nested equilibrium* problems!**

Generalized Feedback Nash Equilibria

Key idea: enforce that $\gamma^* = \left(\gamma^{1*}, \dots, \gamma^{N*}\right) \in (\Gamma^1 \times \cdots \times \Gamma^N)$ also is an ***equilibrium for all sub-games!***

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re-invokes optimization

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$\forall k \in \{t, \dots, T-1\}$:

$$\underbrace{x_{k+1} \overset{\text{def}}{=} f_k\left(x_k, \left(\tilde{\gamma}^i(x_k, k), \gamma^{\neg i*}(x_k, k)\right)\right)}$$

closed-loop dynamics under $(\tilde{\gamma}^i, \gamma^{\neg i*})$

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generalized constraints

$$\text{s.t.} \quad u_t^i \in \mathcal{K}_t^i(x_t, \gamma^{\neg i*}(x_t, t))$$

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Challenge:
Results in T-stage nested equilibrium problem!

Generalized Feedback Nash Equilibria

Key idea: enforce that $\gamma^* = (\gamma^{1*}, \dots, \gamma^{N*}) \in (\Gamma^1 \times \cdots \times \Gamma^N)$ also is an *equilibrium for all sub-games!*

$$\gamma^{i*}(x_t, t) = u_t^{i*} \in \arg\min_{u_t^i} \sum_{k \in \{t, \dots, T\}} J_k^i\left(x_k, \tilde{\gamma}^i(x_k, k), \gamma^{\neg i*}(x_k, k)\right) + J_T^i(x_T)$$

$$\text{s.t.} \qquad u_t^i \in \mathcal{K}_t^i(x_t, \gamma^{\neg i*}(x_t, t))$$

where:

$$\tilde{\gamma}^i(x, k) \stackrel{\text{def}}{=} \begin{cases} u_t^i, & \text{if } k = t \\ \gamma^{i*}(x, k), & \text{if } k > t \end{cases}$$

$$\forall k \in \{t, \dots, T-1\}:$$

$$x_{k+1} \stackrel{\text{def}}{=} f_k\left(x_k, \left(\tilde{\gamma}^i(x_k, k), \gamma^{\neg i*}(x_k, k)\right)\right)$$

Challenge:

Results in T-stage nested equilibrium problem!

Intractable!*

*Forrest Laine et al. 2023

iLQGames Approximation of Feedback Nash Equilibria

Key idea: Feedback games with ***linear dynamic*** and ***quadratic costs*** ***(LQ-Games)*** have a ***closed-form solution***!*
We can use these to ***iteratively approximate feedback Nash*** solutions to non-LQ games!**

*Başar, Tamer, and Geert Jan Olsder. *Dynamic noncooperative game theory*. 1998.

**D. Fridovich-Keil et al. "Efficient Iterative LQ Approximations for Nonlinear Multi-Player General-Sum Differential Games," *2020.*

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initial strategy

$$\gamma^i \leftarrow \hat{\gamma}^i, \forall i \in [N]$$

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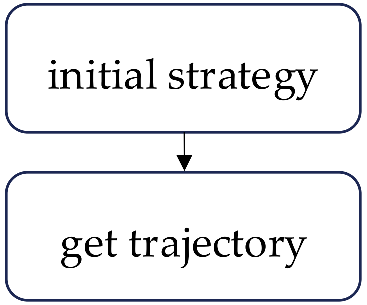

$$x_{t+1} \leftarrow f_t(x_t, \gamma^1(x_t, t), \ldots, \gamma^N(x_t, t)), t \in [T]$$

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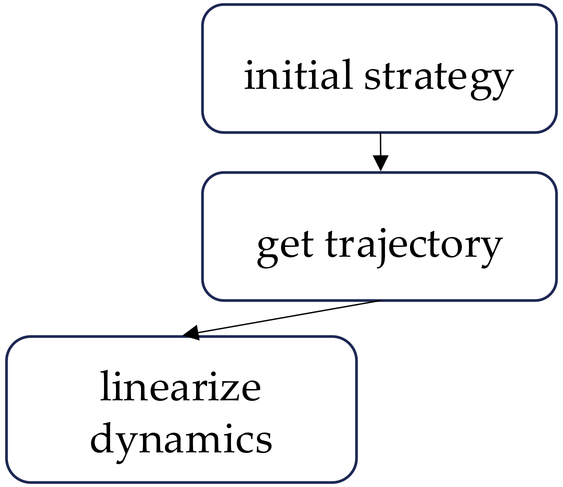

From Taylor-series expansion:

$$\Delta x_{t+1} \approx A_t \Delta x_t + \sum_{i \in [N]} B_t^{i} \Delta u_t^{i}$$

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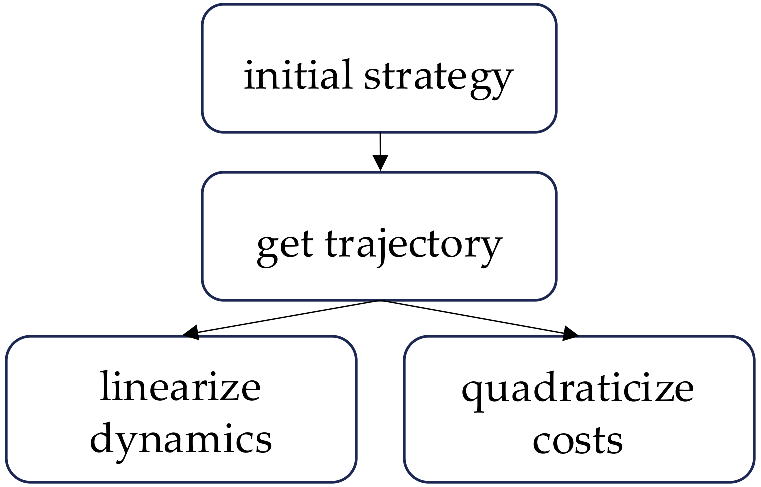

From Taylor-series expansion:

$$J_t^i \approx c + \frac{1}{2}\Delta x_t^\top Q_t \Delta x_t + \frac{1}{2}\sum_{j\in[N]} \Delta u^{j\top} R_t^{ij} \Delta u^j + \Delta u_t^{ij\top} r_t^{ij}$$

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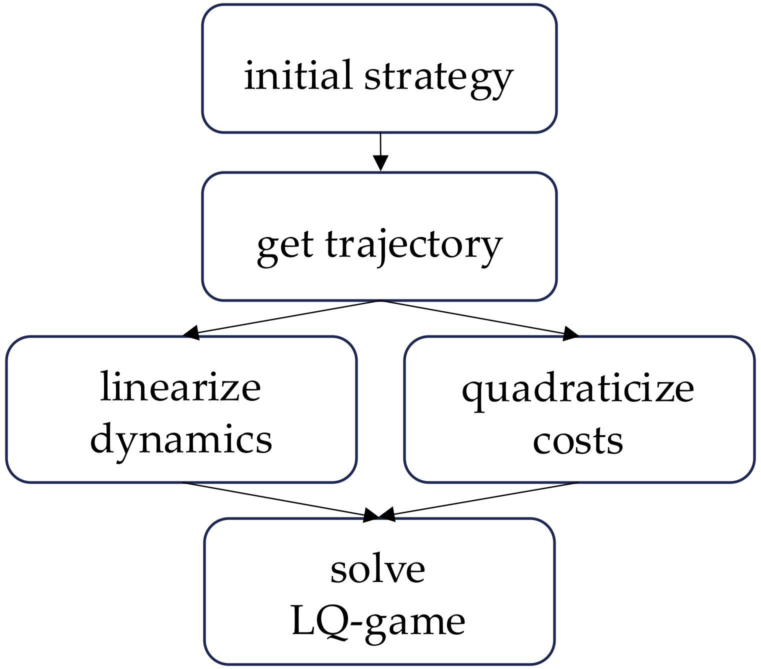

From coupled Riccati equations:

$$\Delta\gamma^i(\Delta x, t) \leftarrow K_t^i \Delta x + \alpha_t^i, \forall i \in [N]$$

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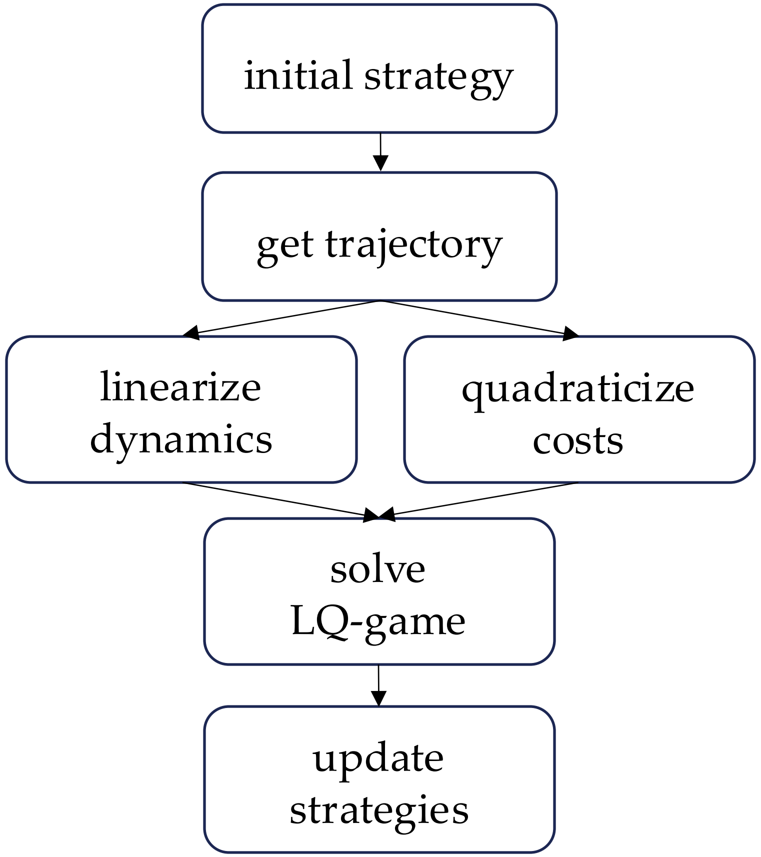

$$\gamma^i \leftarrow \text{stepWithLineSearch}(\gamma^i, \Delta\gamma^i), \forall i \in [N]$$

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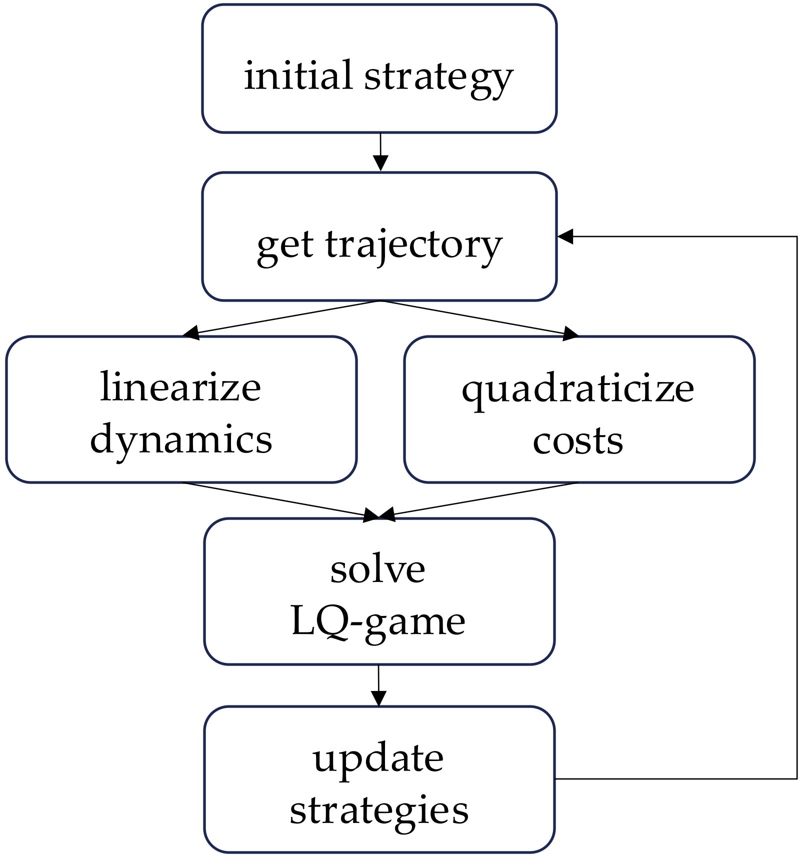

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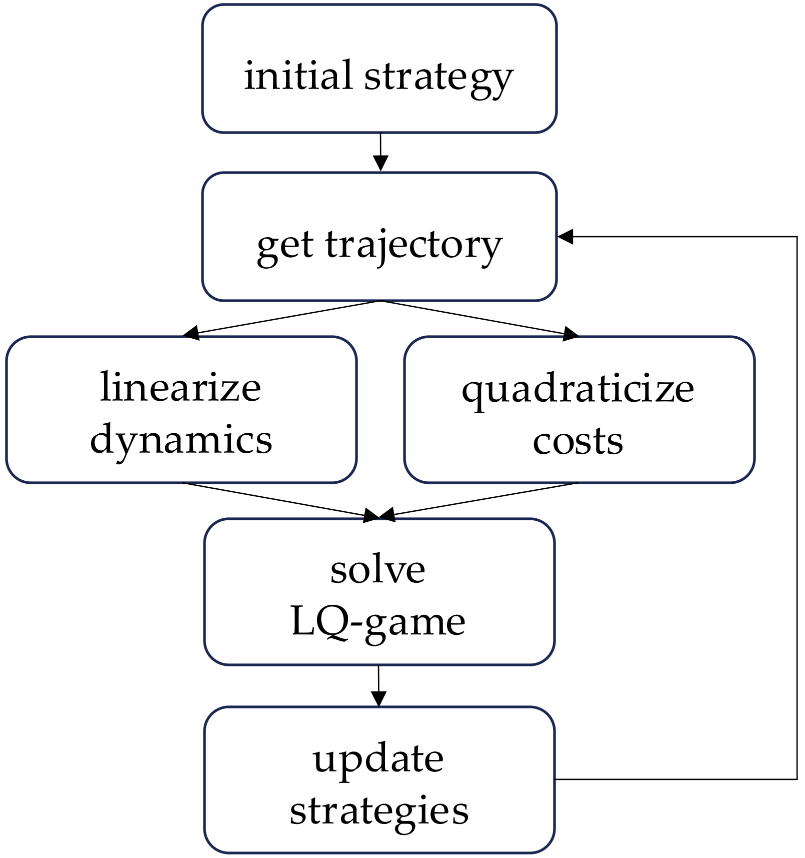

Until convergence!

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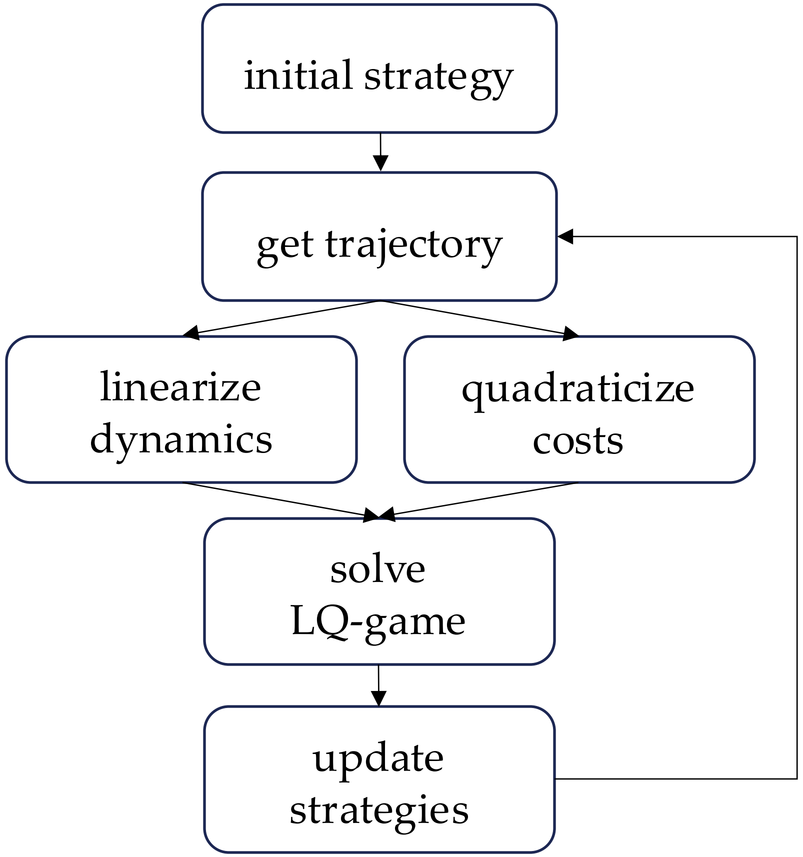
iLQGames | Final Remarks



Limitations:

- Does not handle constraints (extensions exist* but are more complex)
- iLQGames solution is ***not an exact (local) feedback Nash***:
  - **TL;DR:** the solver ignores part of the nested policy gradient!*

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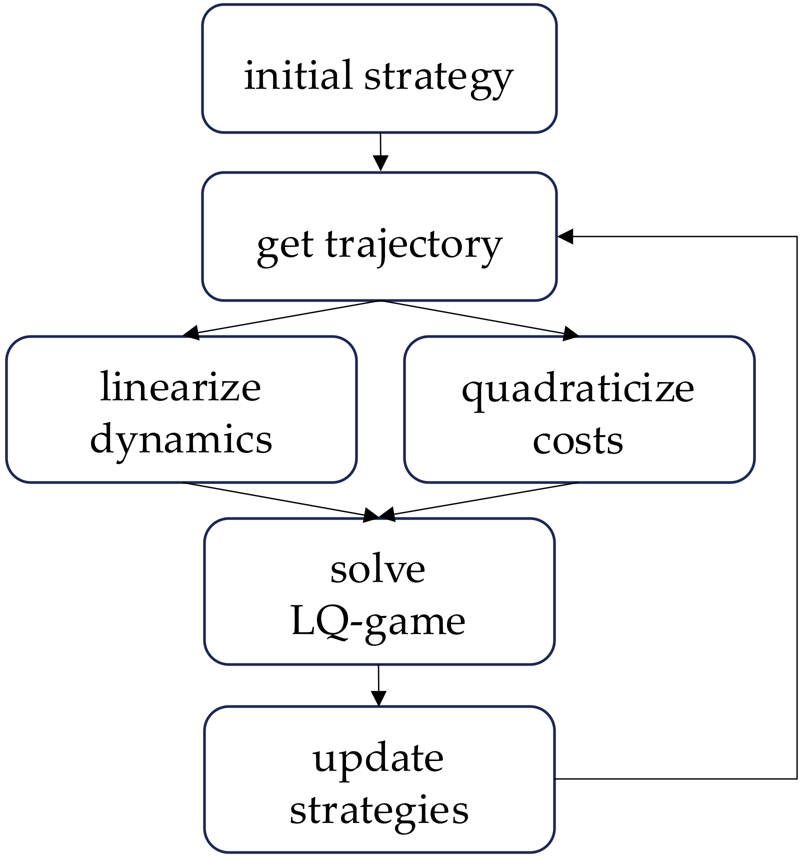

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In Practice:

- Captures characteristics of feedback Nash solutions well
- *Good performance and fast convergence* due to simultaneous updates!

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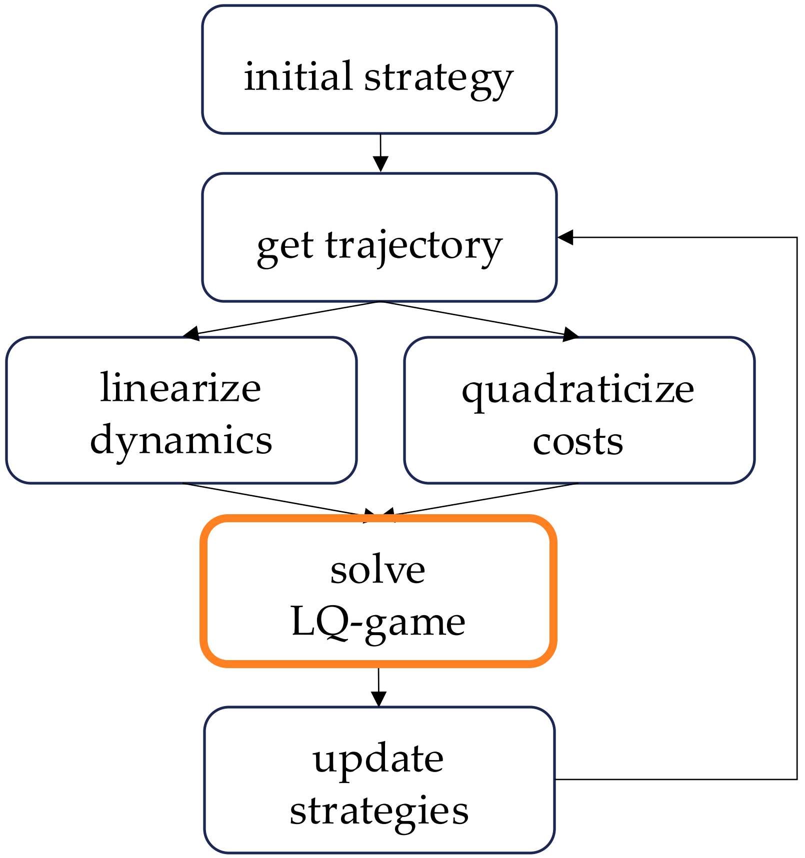

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In Practice:

- Captures characteristics of feedback Nash solutions well
- ***Good performance and fast convergence*** due to simultaneous updates!

Flexibility:

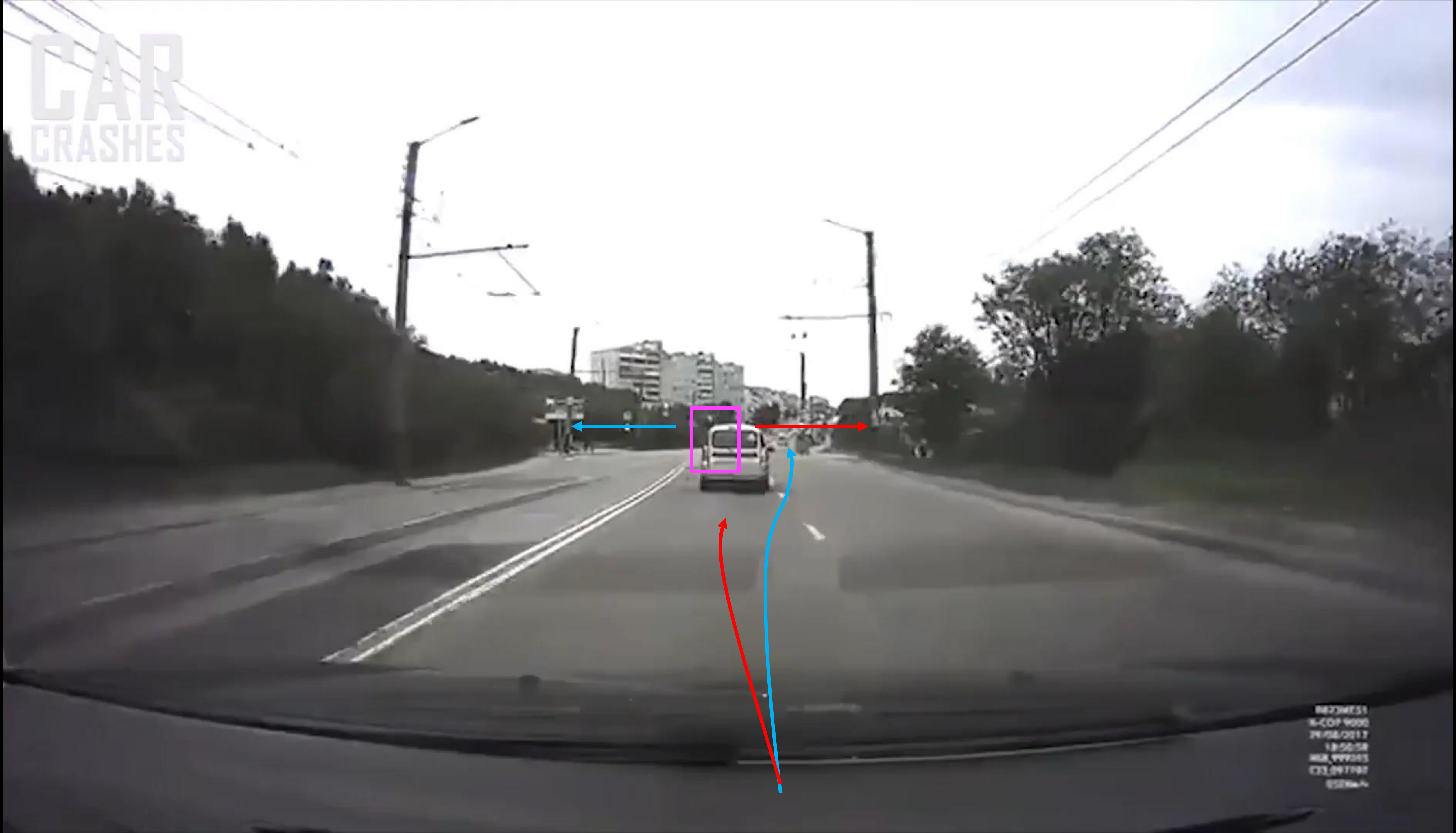
- Extends to ***other equilibrium concepts and information patterns***:
  - open-loop Nash
  - feedback / open-loop Stackelberg equilibria

*Forrest Laine et al. "The computation of approximate generalized feedback nash equilibria." *2023*.

Beyond games with a complete model:
Contingency Games

Joint work with Andrea Bajcsy, Chih-Yuan Chiu, David Fridovich-Keil, Forrest Laine, Laura Ferranti, Javier Alonso-Mora.


CAR
CRASHES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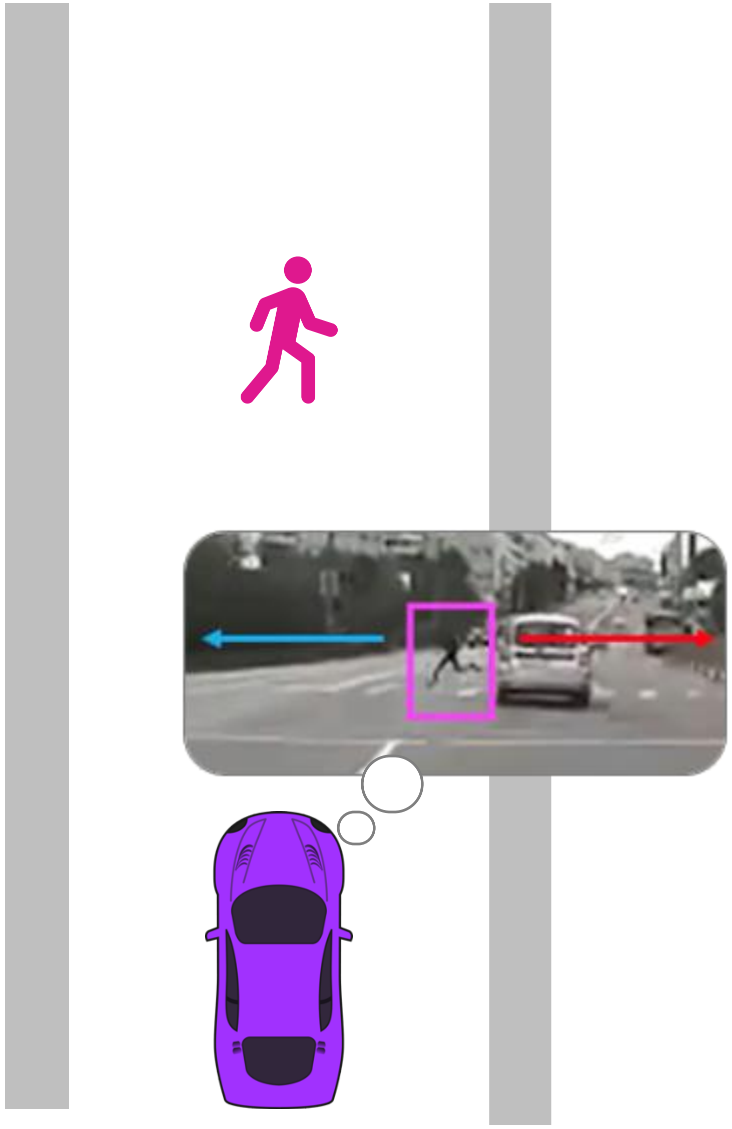
Challenge: **intents** are *not* known a priori

$$\left.\begin{aligned} \tau^{i*} &\in \arg\min_{\tau^i} J^i(\tau^i, \tau^{\neg i}) \\ \text{s.t.}\quad \tau^i &\in \mathcal{K}^i(\tau^{\neg i}) \end{aligned}\right\} \quad i \in \{H, R\}$$

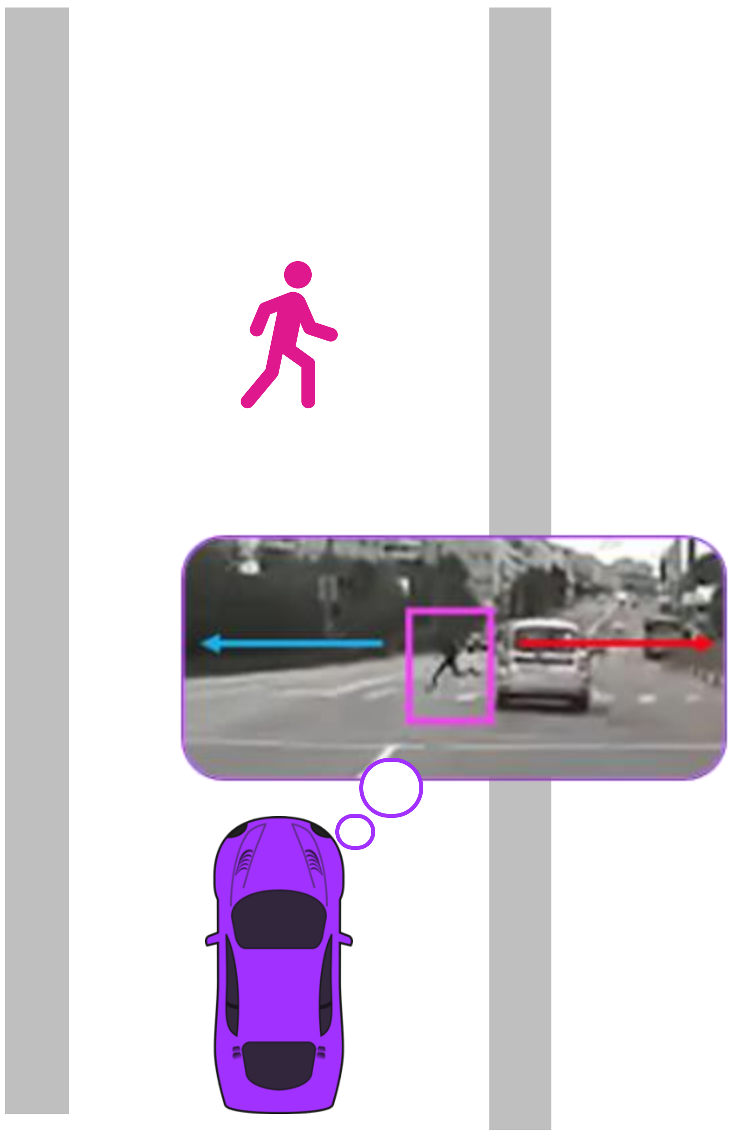


$$\left.\begin{aligned} \tau^{i*} &\in \arg\min_{\tau^i} J^i(\tau^i, \tau^{\neg i}; \theta) \\ \text{s.t.}\quad \tau^i &\in \mathcal{K}^i(\tau^{\neg i}; \theta) \end{aligned}\right\} \quad i \in \{H, R\}$$

Maintain belief over **intent parameters**:

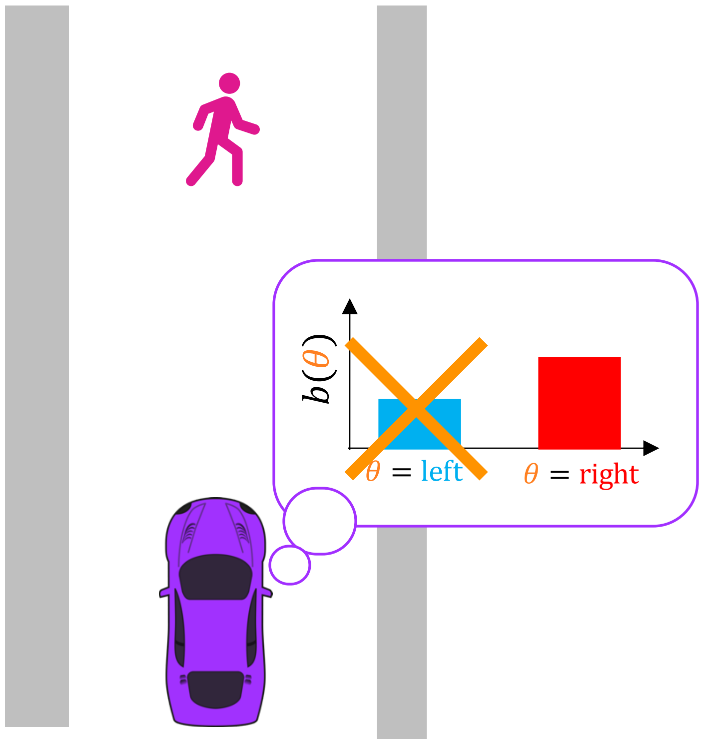
$$b_t(\theta) := P(\theta \mid z_{0:t}) \quad \begin{cases} \textit{e.g.,} \\ \text{Particle Filter} \\ \text{UKF} \end{cases}$$

Teaser: David Fridovich-Keil will show you how to do this in week 9!



Certainty-Equivalent

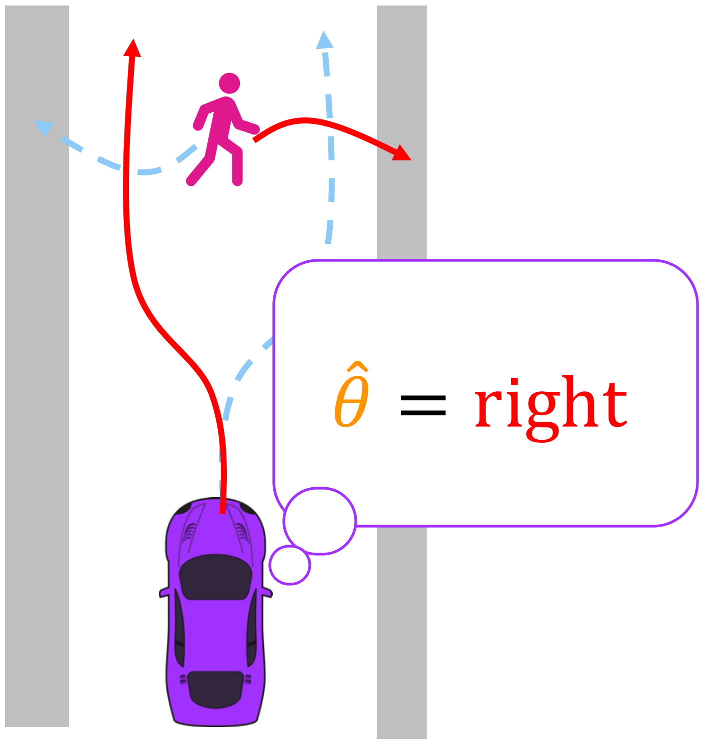
$$\hat{\theta} = \arg\max_{\theta \in \Theta} b(\theta)$$



[Liu 2022, Mehr 2023, Schwarting 2019, Sadigh 2016]

Certainty-Equivalent

$$\arg\min_{\tau^i} J^i(\tau^i, \tau^{\neg i}; \hat{\theta})$$
$$\text{s.t.} \quad \tau^i \in \mathcal{K}^i(\tau^{\neg i}; \hat{\theta})$$



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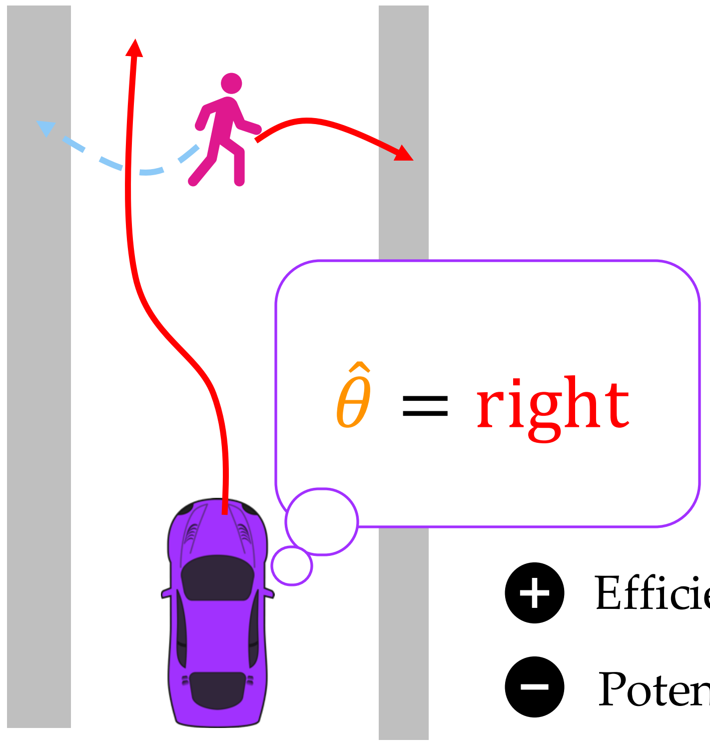

- ⊕ Efficient solvers
- ⊖ Potentially unsafe

[Liu 2022, Mehr 2023, Schwarting 2019, Sadigh 2016]

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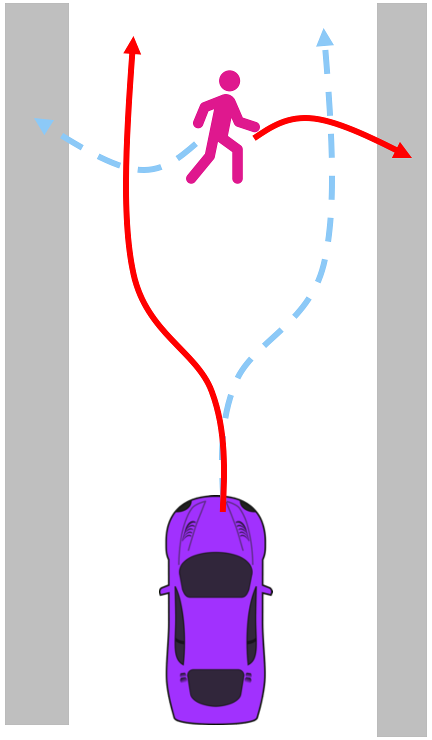

- Efficient solvers
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[Liu 2022, Mehr 2023, Schwarting 2019, Sadigh 2016]

Fixed Uncertainty

Robot

$$\arg\min_{\tau^R} \mathbb{E}_{\theta\sim b}\left[J^R(\tau^R, \tau^H_\theta; \theta)\right]$$
$$\text{s.t.} \quad \tau^R \in \mathcal{K}^R(\tau^H_\theta; \theta)$$

Human
w/ intent θ

$$\arg\min_{\tau^H_\theta} J^H(\tau^H_\theta, \tau^R_\theta; \theta)$$
$$\text{s.t.} \quad \tau^H_\theta \in \mathcal{K}^H(\tau^R_\theta; \theta)$$

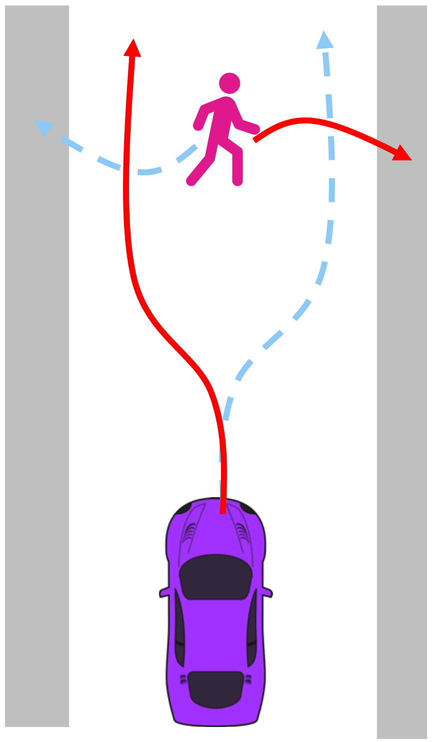
[Laine 2021, Le Cleac'h 2021]

Certainty-Equivalent

$$\arg\min_{\tau^i} J^i(\tau^i, \tau^{\neg i}; \hat{\theta})$$
$$\text{s.t.} \quad \tau^i \in \mathcal{K}^i(\tau^{\neg i}; \hat{\theta})$$

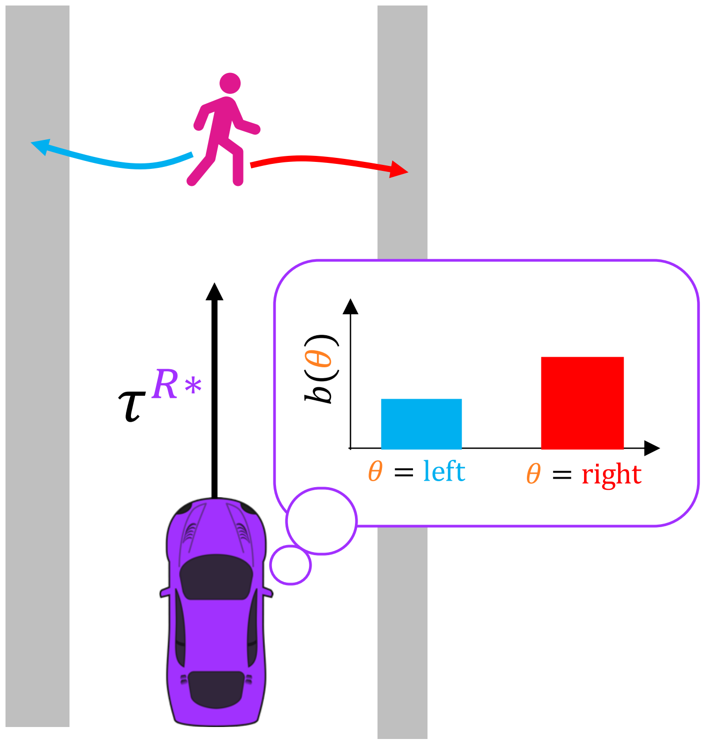
Fixed Uncertainty

$$\arg\min_{\tau^R} \mathbb{E}_{\theta \sim b}\left[J^R(\tau^R, \tau^H_\theta; \theta)\right]$$
$$\text{s.t.} \quad \tau^R \in \mathcal{K}^R(\tau^H_\theta; \theta)$$



- (+) Efficient solvers
- (−) Potentially unsafe

- (+) Accounts for uncertainty
- (−) No future info gain; conservative!



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[Laine 2021]

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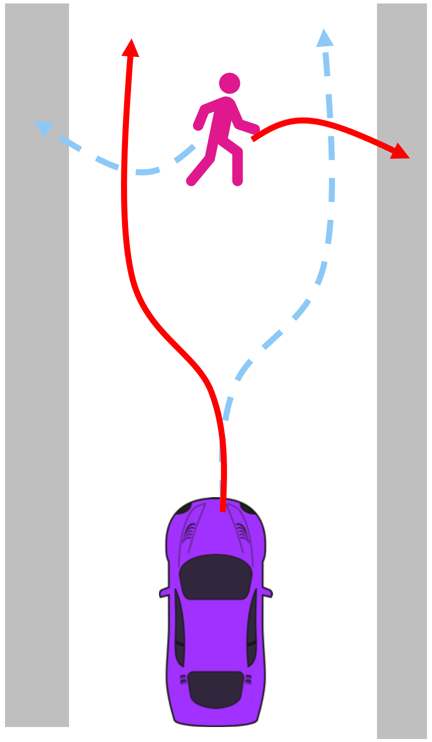

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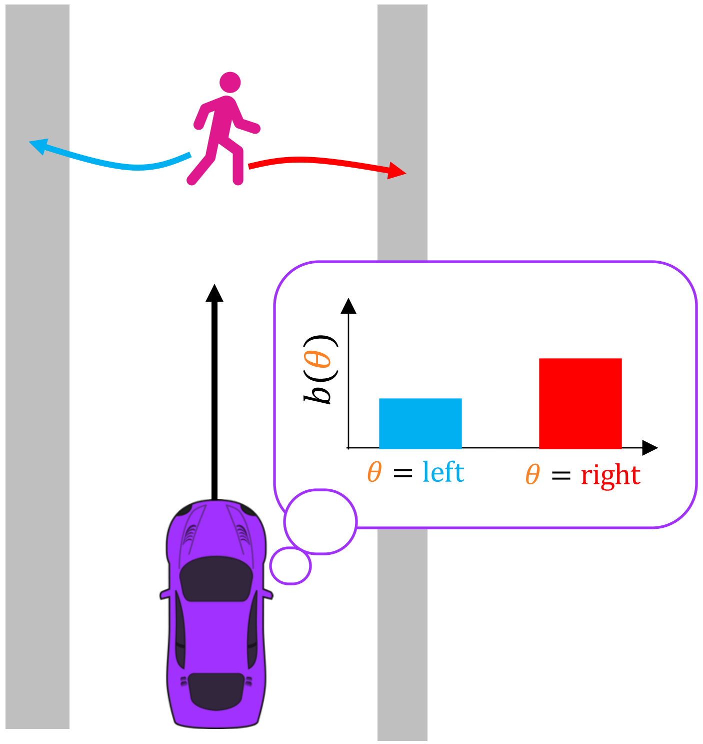
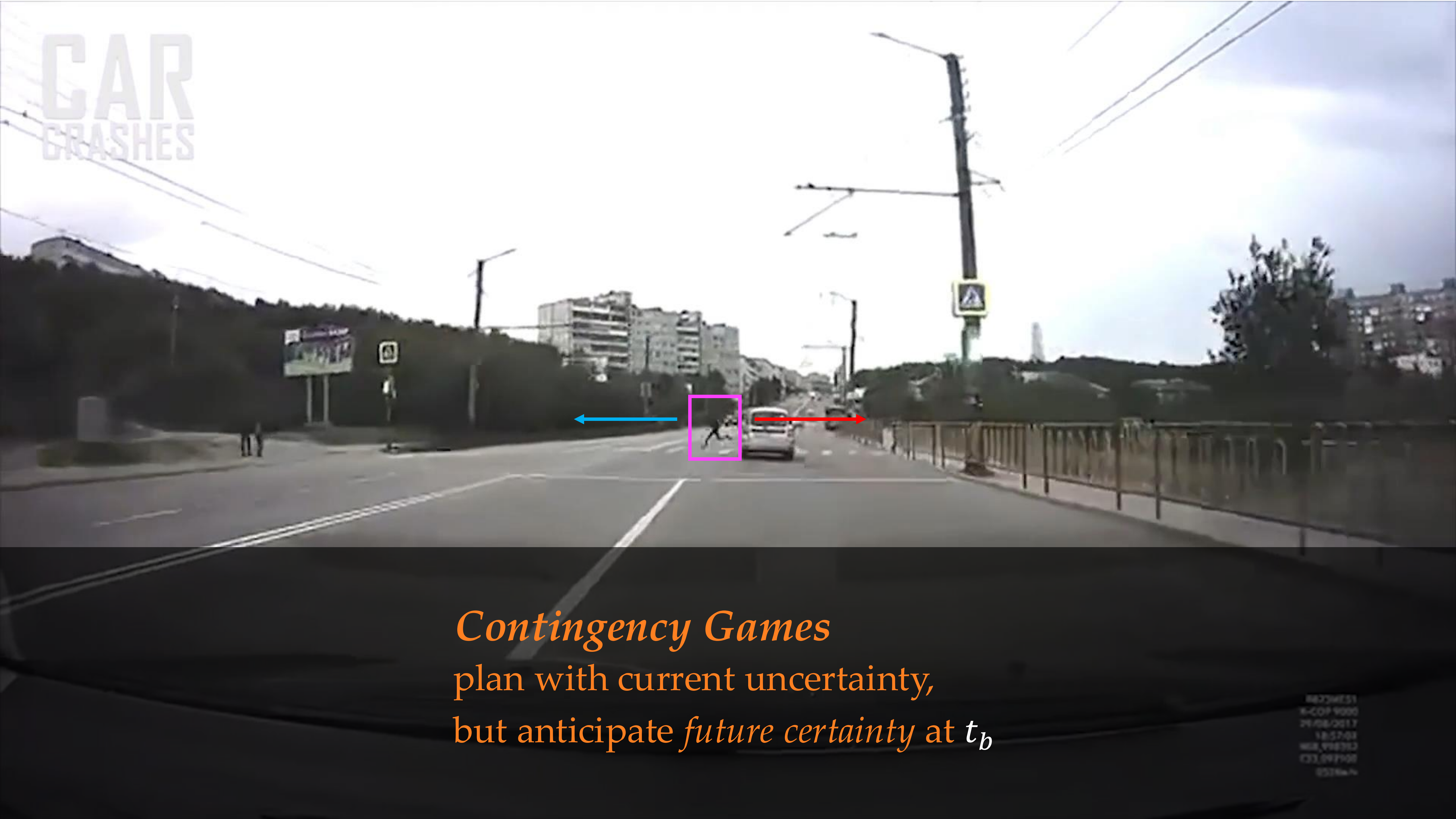


Contingency Games

Bridge the gap by *accounting for future information*
while preserving *tractability*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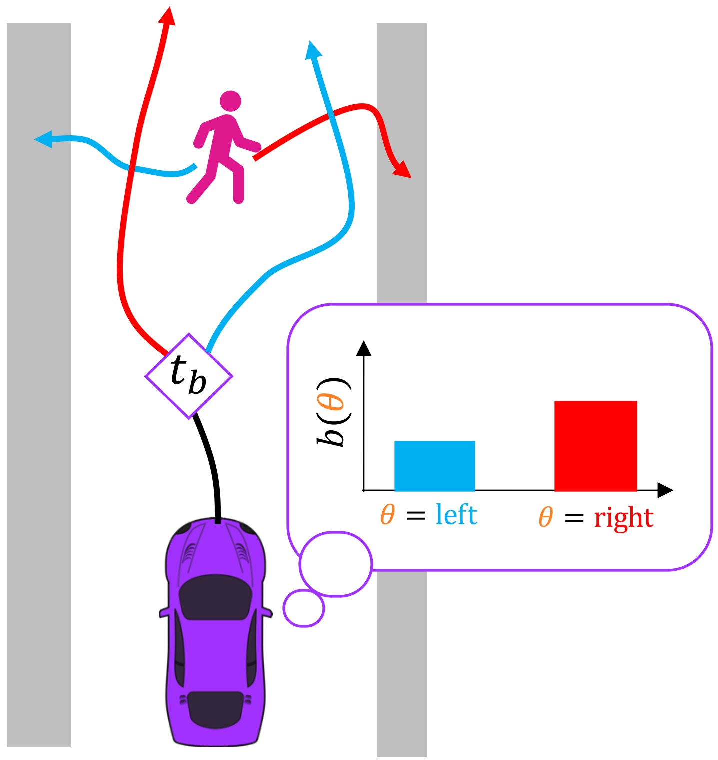
Contingency Games
plan with current uncertainty,
but anticipate *future certainty* at t_b



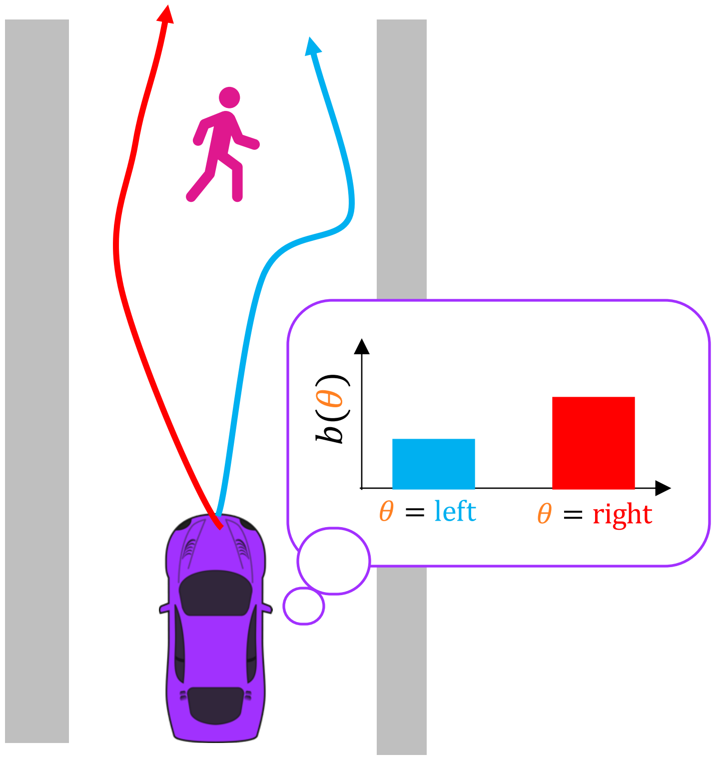
Contingency Games

plan with current uncertainty,

but anticipate *future certainty* at t_b

before t_b : a single plan that considers *all hypotheses*

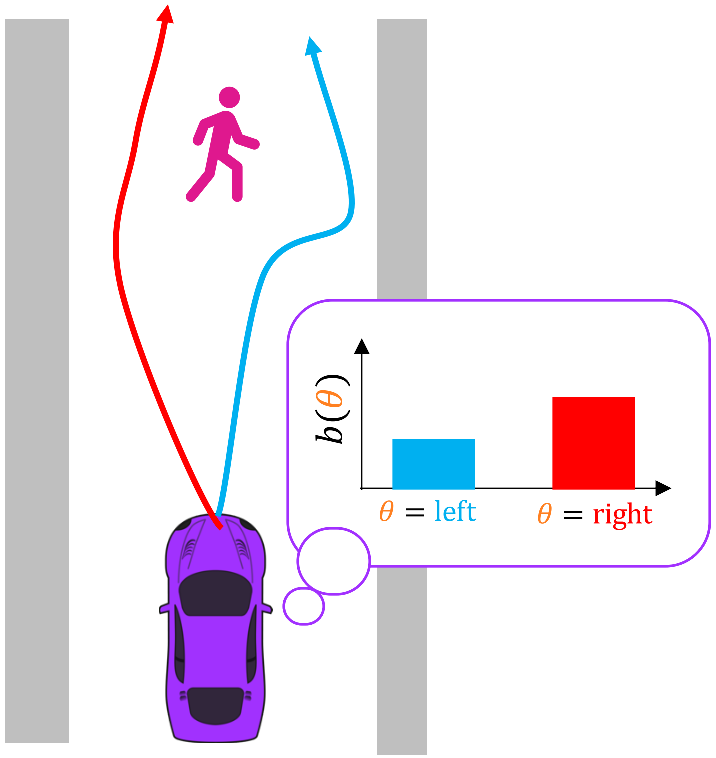
after t_b separate plans conditioned on each outcome



Contingency Games

Robot $\arg\min_{\tau^R_\Theta} \mathbb{E}_{\theta\sim b}\left[J^R(\tau^R_\theta, \tau^H_\theta; \theta)\right]$

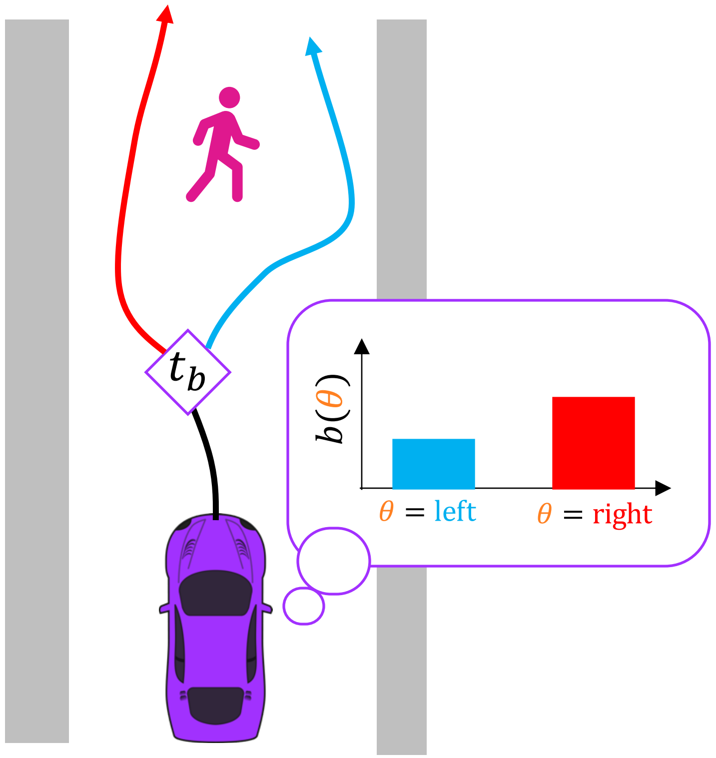
Keeps separate plan for each of the $|\Theta|$ hypotheses



Contingency Games

Robot

$$\begin{cases} \arg\min_{\tau^R_\Theta} \mathbb{E}_{\theta \sim b}\left[J^R(\tau^R_\theta, \tau^H_\theta; \theta)\right] \\ \text{s.t.} \quad \tau^R_\theta \in \mathcal{K}^R(\tau^H_\theta; \theta) \end{cases}$$

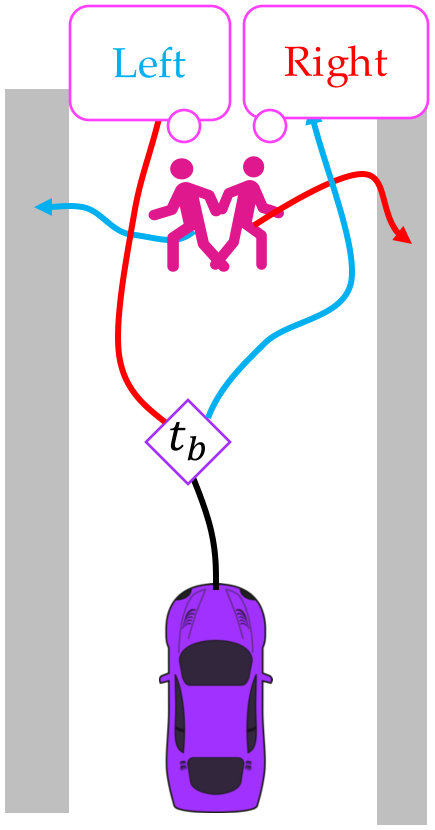


Contingency Games

Robot

$$\begin{cases} \underset{\tau^R_\Theta}{\arg\min}\ \mathbb{E}_{\theta\sim b}\left[J^R(\tau^R_\theta,\tau^H_\theta;\theta)\right] \\ \text{s.t.}\quad \tau^R_\theta \in \mathcal{K}^R(\tau^H_\theta;\theta) \\ \qquad \tau^R_{\theta^j}(t) \equiv \tau^R_{\theta^k}(t) \\ \qquad \forall\left(t,\theta^j,\theta^k\right) \in ([0, t_b] \times \Theta^2) \end{cases}$$

All plans must be consistent up to t_b



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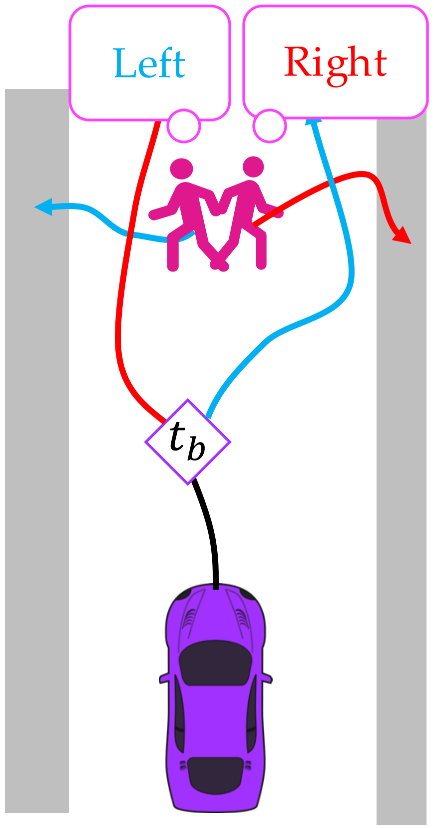
Human
w/ intent $\theta \in \Theta$

$$\begin{cases} \arg\min_{\tau^H_\theta} J^H(\tau^H_\theta, \tau^R_\theta; \theta) \\ \text{s.t.} \quad \tau^H_\theta \in \mathcal{K}^H(\tau^R_\theta; \theta) \end{cases}$$



Contingency Games

$$\tau_{\Theta}^{R*} = \arg\min_{\tau_{\Theta}^{R}} \mathbb{E}_{\theta \sim b}\left[J^{R}(\tau_{\theta}^{R}, \tau_{\theta}^{H}; \theta)\right]$$

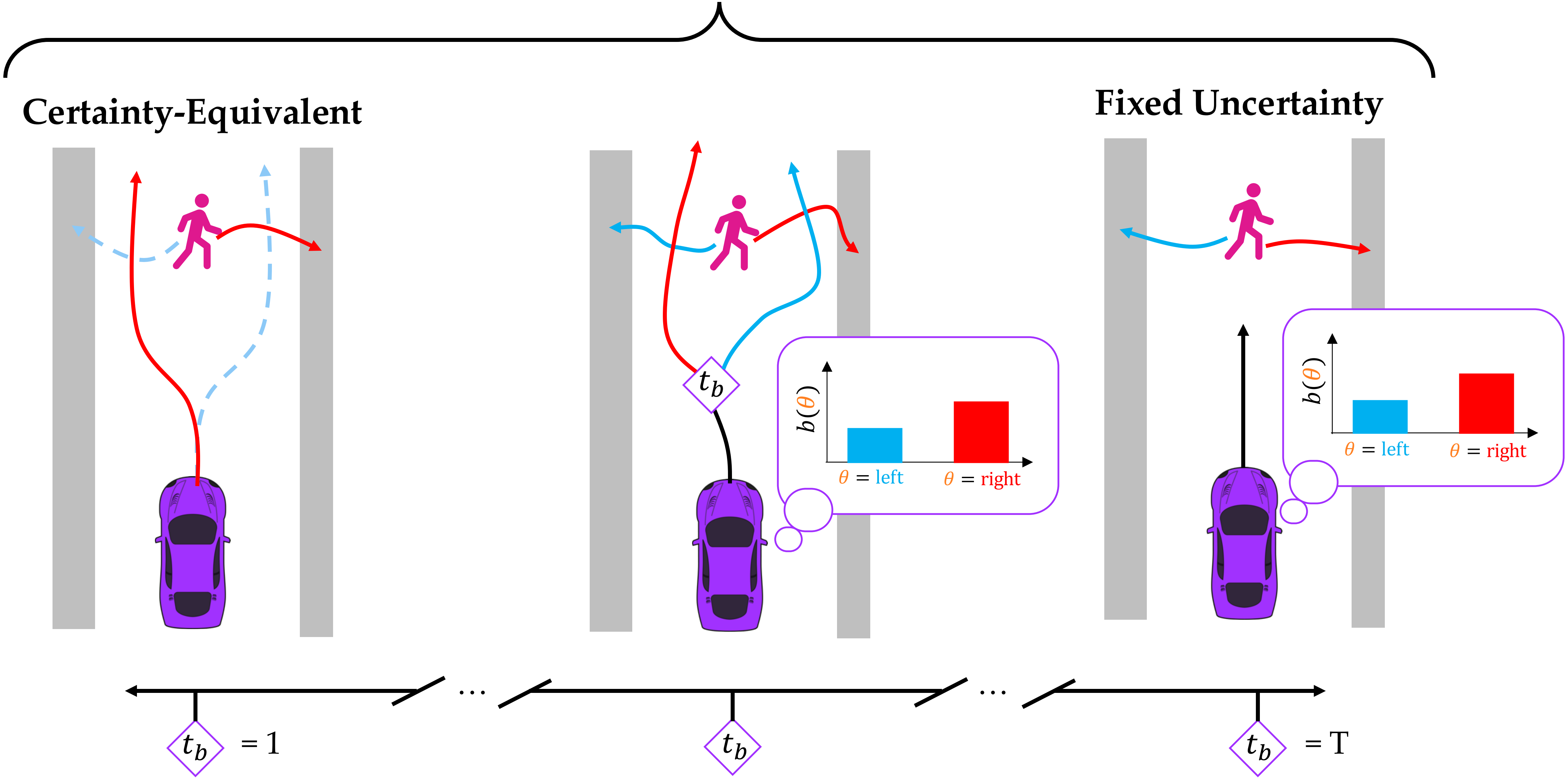
$$\text{s.t.} \quad \tau_{\theta}^{R} \in \mathcal{K}^{R}(\tau_{\theta}^{H}; \theta)$$

$$\tau_{\theta^j}^{R}(t) \equiv \tau_{\theta^k}^{R}(t) \ \forall \left(t, \theta^j, \theta^k\right) \in ([0, t_b] \times \Theta^2)$$

$$\left.\begin{aligned} \tau_{\theta}^{H*} &= \arg\min_{\tau_{\theta}^{H}} J^{H}(\tau_{\theta}^{H}, \tau_{\theta}^{R}; \theta) \\ &\text{s.t.} \quad \tau_{\theta}^{H} \in \mathcal{K}^{H}(\tau_{\theta}^{R}; \theta) \end{aligned}\right\} \forall \theta \in \Theta$$

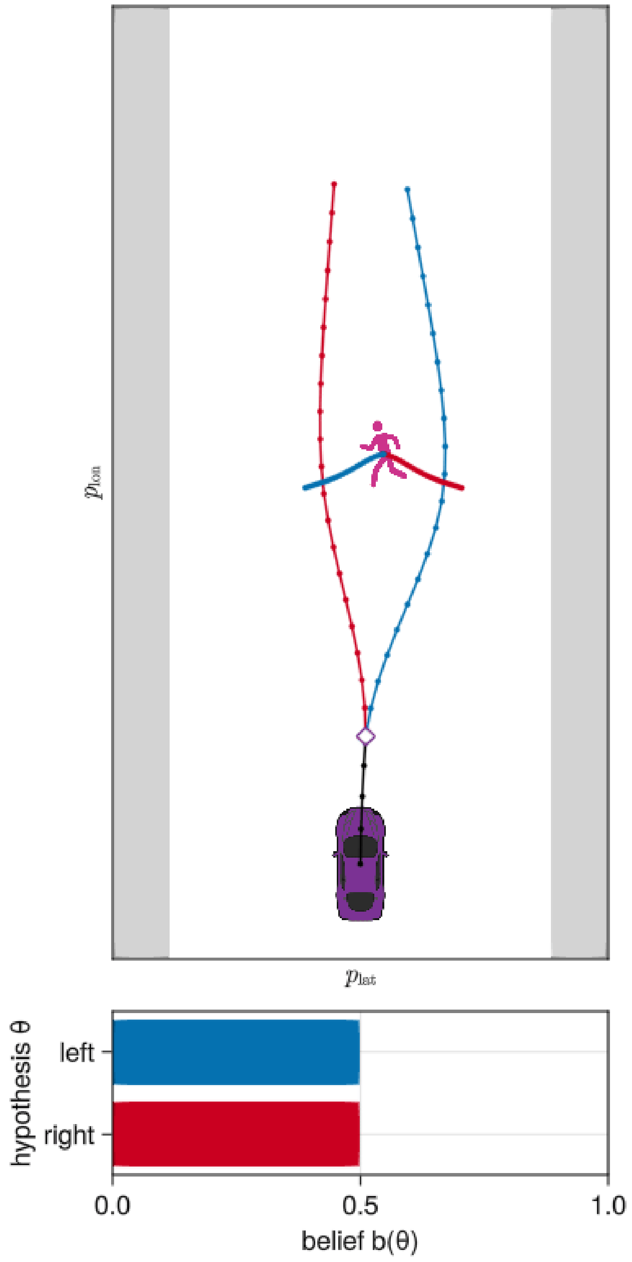
Demo

Spectrum of *Contingency Games*



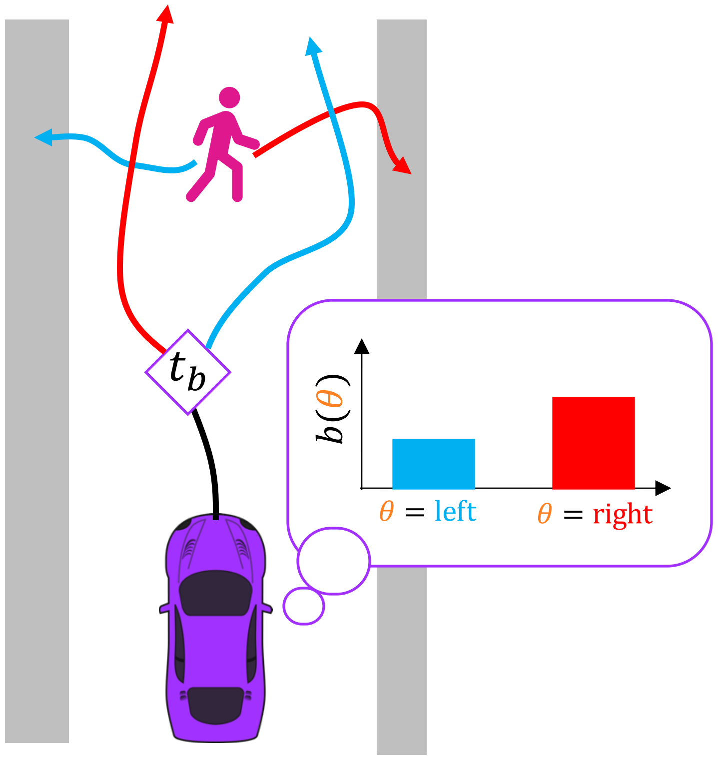
Branching Time *(t_b): known, tunable* parameter*

**e.g., [Dvro 2021, Bajcsy 2021]*



Receding-horizon online operation

By estimating the belief and branching time online, we obtain an *adaptive game-theoretic motion planner.*



Key Result

Contingency games generate more *efficient* plans than fixed-uncertainty games at comparable levels of *safety*.

lasse-peters.net/pub/contingency-games

Main Take-Aways

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- Dynamic games capture interaction via ***coupled optimization***

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- Dynamic games capture interaction via *coupled optimization*
- *Open-loop generalized Nash equilibrium* problems can be solved with *Mixed-complementary solvers*
- Feedback games result in nested equilibrium problems → hard to solve exactly
- *iLQGames* tractably approximate *feedback Nash solutions* and other equilibrium concepts (e.g. Stackelberg, open-loop Nash)

Main Take-Aways

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- *Open-loop generalized Nash equilibrium* problems can be solved with *Mixed-complementary solvers*
- Feedback games result in nested equilibrium problems → hard to solve exactly
- *iLQGames* tractably approximate *feedback Nash solutions* and other equilibrium concepts (e.g. Stackelberg, open-loop Nash)
- *Contingency Games* efficiently capture uncertainty in games by modeling a future time at which uncertainty will resolve

Game-Theoretic Models
for Multi-Agent Interaction

Lasse Peters

Find game solvers, modeling infrastructure and more at

github.com/JuliaGameTheoreticPlanning
github.com/lassepe

A Naïve Formulation of Games over Feedback Strategies

As before, but now with *decision variables in the space of time-varying feedback strategies*: $\Gamma^i \ni \gamma^i : \mathcal{X} \times [T] \to \mathcal{U}^i$

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i \in [N] \quad \begin{cases} \min\limits_{\gamma^i \in \Gamma^i} & J^i(\gamma^i, \gamma^{\neg i}) \\ \text{s.t.} & \gamma^i \in \mathcal{K}^i(\gamma^{\neg i}) \end{cases}
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Problem: solutions of this problem *may not make use of feedback* in a meaningful way!

Can show: original open-loop Nash solutions also satisfy this! $\left((x, t) \mapsto u_t^i\right) \in \Gamma^i$

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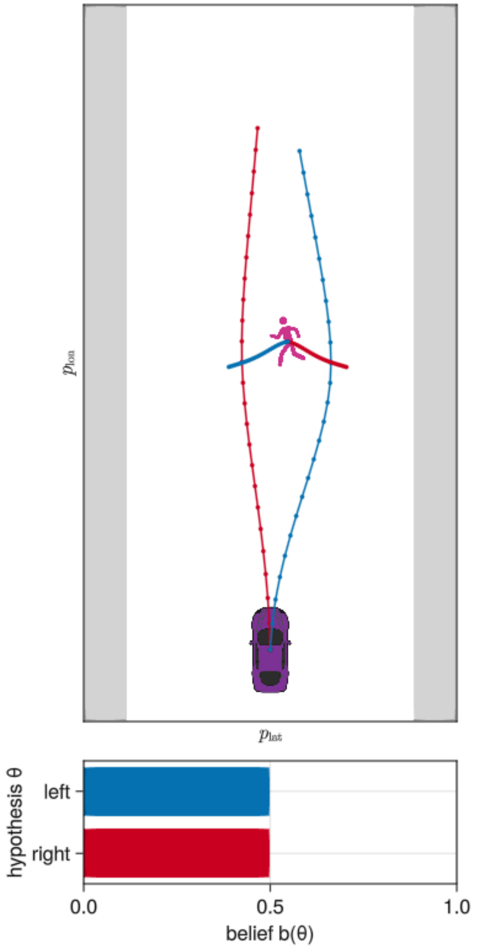
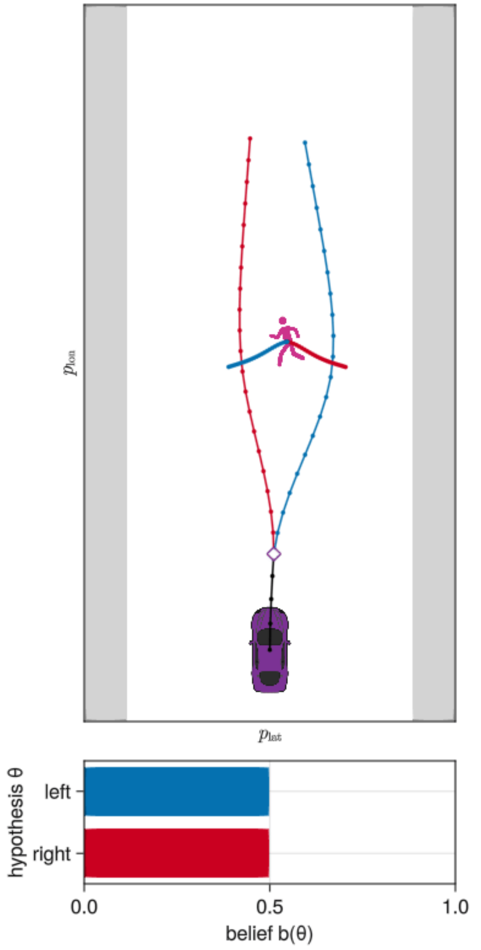
Qualitative Results

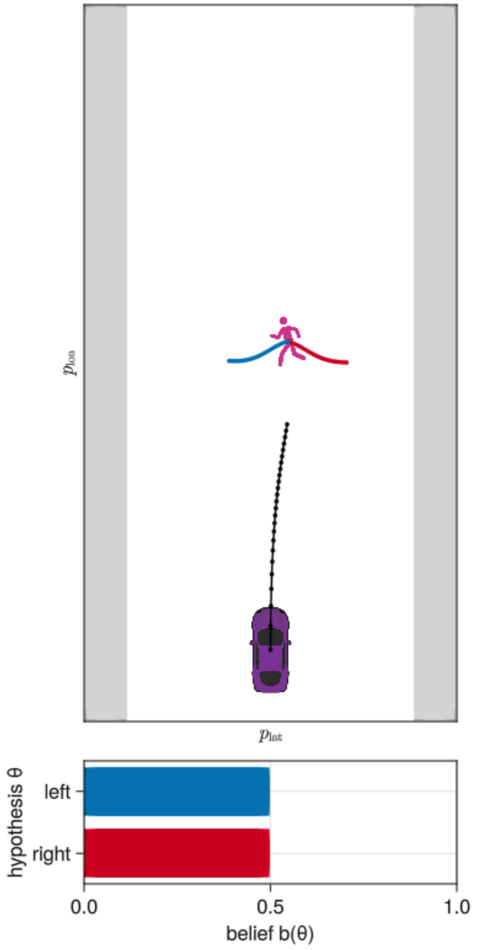
Baseline 1
Certainty-Equivalent

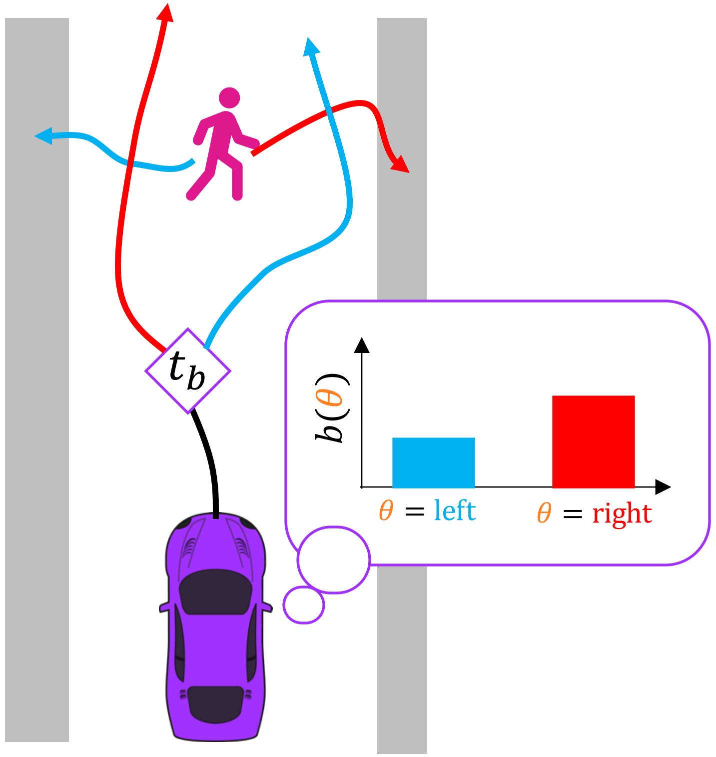


Contingency Games



Baseline 2
Fixed Uncertainty





Solving *Contingency Games*

- Formulate KKT conditions
- KKT system is a *mixed complementarity prob.*
- Reformulate and use off-the-shelf solvers*
- Find satisfying trajectories $\left(\tau_{\Theta}^{R*}, \tau_{\theta^1}^{H*}, \ldots, \tau_{\theta^{|\Theta|}}^{H*}\right)$

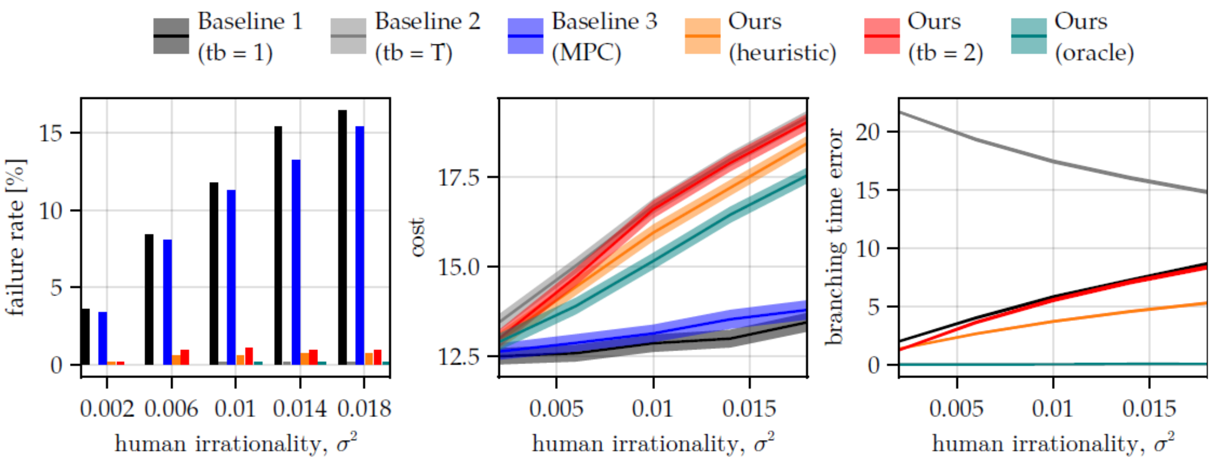
Example: 3-player game, 25 time steps, 2 hypotheses

3,208 decision variables, solution in **35 ms**

* [Dirkse 1995]

Quantitative Results

Jaywalking

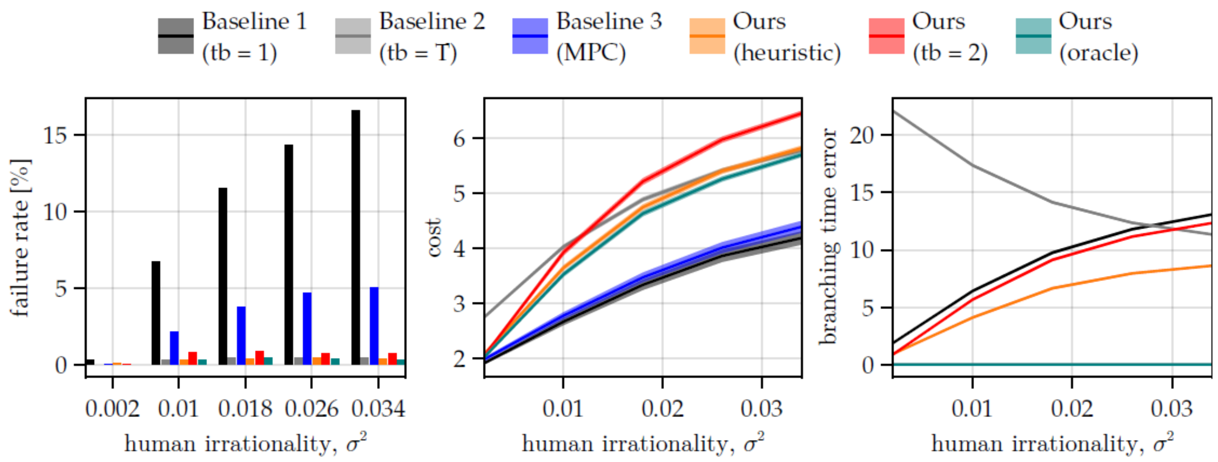


(a) Failure rate

(b) Closed-loop performance for successful runs

(c) Branching time error

Overtaking



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