Last Time:

- ☐ decision-making
- ☐ MDPs

Lecture 3

HRI, Fall '24

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Today:

- ☐ values & Q-values
- ☐ hands-on exercise

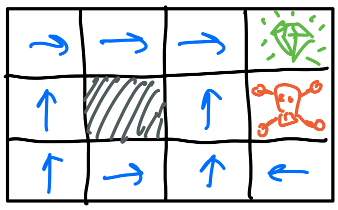
- Today: MDPs
- Thurs: Probability
- Next Tues: Paper Reading (Fundamentals)
- Next Thurs: —"— (Math. Human Models)

RECAP: We left off having defined a Markov Decision Process:

- set of states S
- set of actions A
- transition function $P(s'|s,a)$ $\leftarrow$ {or $T(s',a,s)$}
- reward function $r(s,a)$ // sometimes $r(s)$, $r(s,a,s')$, $r(s,a)$
- discount factor $\gamma \in [0,1]$

MDP quantities we have seen so far:

- policy [π]: choice of action for each state



- cumulative reward: sum of (discounted) rewards

Our goal is to find a policy π that maximizes the discounted sum of rewards:

$$\overset{1}{\gamma^0} r(s^0,a^0) + \gamma^1 r(s^1,a^1) + \gamma^2 r(s^2,a^2) + \dots$$

this trades off short & long-horizon reward

VALUE FUNCTION

lets quantify the best cumulative reward we could ever get starting from a state s. This will precisely be our value function: $V: S \to \mathbb{R}$

Simpler case: Deterministic Transition.

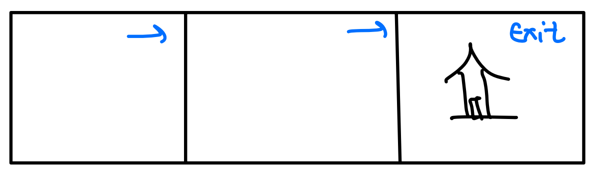
$$V(s) = \underbrace{\max_{\pi}}_{\text{② try every possible policy and get the maximum possible value}} \underbrace{\left[\sum_{t=0}^{\infty} \gamma^t r(s^t, a^t) \right]}_{\text{① discounted sum of rewards}}$$

It turns out that we can rewrite this equation recursively because of Bellman's "principle of optimality." It says that you can decompose a complex problem into smaller sub-problems. Mathematically, it lets us redefine our value:

$$V(s) = \underbrace{\max_{a \in \mathcal{A}} \Big[\underbrace{r(s,a)}_{\text{reward right now!}} + \gamma \cdot V(s') \Big]}_{\text{maximum possible discounted value of next state plus current reward.}}$$

s' is the next state you get to after taking a from state s

ex.



s= 0 1 2

$\mathcal{A} = \{ \rightarrow, \text{Exit} \}$

you can take Exit @ home (s=2)

deterministic $P(s' \mid s, a) = \begin{cases} 1 & \text{if } s' = s + a \\ 0 & \text{else} \end{cases}$

$r(s,a) = 0$ UNLESS you $r(s=2, a=\text{Exit}) = +1$

① $V(s=2) = r(s=2, a=\text{Exit}) = \boxed{+1}$

② $V(s=1) = \underbrace{r(s=1, a=\rightarrow)}_{=0} + \gamma \cdot V(s=2) = \boxed{\gamma}$

③ $V(s=0) = \underbrace{r(s=0, a=\rightarrow)}_{=0} + \gamma \cdot V(s=1) = \boxed{\gamma^2}$

Stochastic Transitions

$$V(s) = \max_{a \in \mathcal{A}} \Big[r(s,a) + \gamma \underbrace{\sum_{s' \in S} P(s'|s,a) V(s')}_{} \Big]$$

when transitions are stochastic, we get expected value across next states we may visit.

$\boxed{\text{Q-function}}$

One more helpful quantity is the Q-function (Q value). You'll see this alot in papers :). Expected value of taking some action a from state s, and then acting optimally from there on.

$\boxed{f(s,a) = s'}$

Determistic: $Q(s,a) = r(s,a) + \gamma \cdot V(s')$ ← taking a

Stochastic: $Q(s,a) = r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V(s')$

Intuition: How good is action a in the short & long-term?

Optimal policy: $\pi(s) = \arg\max_{a \in A} Q(s,a)$

But, we were staring at our value function, we need to deal w/ the recursion!

$$V(s) = \max_{a \in \mathcal{A}} \Big[r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V(s') \Big]$$

$$V(s') = \max_{a \in \mathcal{A}} \Big[r(s',a) + \gamma \sum_{s'' \in S} P(s''|s',a) V(s'') \Big]$$

recursive ("dynamic programming")

VALUE ITERATION

$V[s] \leftarrow 0$ // current estimate of value (zero $\forall s \in S$!)

$V'[s] \leftarrow 0$ // next / updated estimate of value

$\delta \leftarrow 0$ // maximum change btwn. V and V'

while True:

$V \leftarrow V'$ and $\delta \leftarrow 0$ // we will update V' so remember prior values

for each state $s \in S$:

$$V'[s] \leftarrow \max_{a \in A} \left[r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V[s'] \right]$$

if $|V'[s] - V[s]| > \delta$

$\delta \leftarrow |V'[s] - V[s]|$

if $\delta \leq \varepsilon$
return $V'[s]$

// stopping condition: ε = max error
if we have less error than ε, return!

Exercise [from CS188 Berkeley]

Pacman is using MDPs to maximize his expected utility. In each environment:

- deterministic transition ↑ ← ↓ →
- Pacman has the standard actions {North, East, South, West} unless blocked by an outer wall
- There is a reward of 1 point when eating the dot (for example, in the grid below, $R(C, South, F) = 1$)
  i.e. transition to state F
- The game ends when the dot is eaten
  i.e. zero reward at F state

(a) Consider a the following grid where there is a single food pellet in the bottom right corner (F). The **discount** factor is 0.5. There is no living reward. The states are simply the grid locations.

i.e. you only get reward at the dot

A	B	C
D	E	F ○

(i) What is the optimal policy for each state?

(ii) What is the optimal value for the state of being in the upper left corner (A)? Reminder: the discount factor is 0.5.

Run 4 iterations of value iteration by filling out the table:

iter	V(A)	V(B)	V(C)	V(D)	V(E)	V(F)
0	0	0	0	0	0	0
1						
2						
3						
4						

Solution:

Pacman is using MDPs to maximize his expected utility. In each environment:

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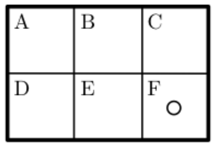

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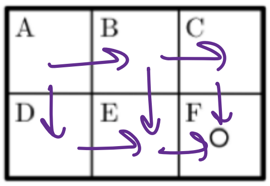

(ii) What is the optimal value for the state of being in the upper left corner (A)? Reminder: the discount factor is 0.5.

$V^*(A) = 0.25$

k	V(A)	V(B)	V(C)	V(D)	V(E)	V(F)
0	0	0	0	0	0	0
1	0	0	1	0	1	0
2	0	0.5	1	0.5	1	0
3	0.25	0.5	1	0.5	1	0
4	0.25	0.5	1	0.5	1	0

$$k=1: \; V(E) = \max\big[r(E,\rightarrow) + 0.5\, V(F),\; r(E,\uparrow) + 0.5\, V(B),\; r(E,\leftarrow) + 0.5\, V(D) \big] = \max[1, 0, 0] = \boxed{1}$$

$$k=2: \; V(B) = \max\big[r(B,\rightarrow) + 0.5\, V(C),\; r(B,\downarrow) + 0.5\, V(E),\; r(B,\leftarrow) + 0.5\, V(A) \big] = \max[0.5, 0.5, 0] = \boxed{0.5}$$

$$k=3: \; V(A) = \max\big[r(A,\rightarrow) + 0.5\, V(B),\; r(A,\downarrow) + 0.5\, V(D) \big] = \max[0.25, 0.25] = \boxed{0.25}$$

If I give you $v^*(s)$ optimal value, then I can extract the optimal policy via a ONESTEP DECISION-MAKING PROBLEM!

$$\pi^*(s) = \arg\max_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s'} P(s'|s,a)\, v^*(s') \right]$$