Lecture 2
IR, Spring '24
Andrea Bajcsy

01/22/24

Last Time:

- [ ] Interaction as feedback loops

This Time:

- [ ] basics of dynamical systems & control theory
- [ ] need for safety analysis
- [ ] models of uncertainty

Dynamical systems & control theory provides us with a set of powerful mathematical frameworks for describing complex

1) (embodied) systems
2) (optimal) behavior
3) feedback loops

One such framework that is particularly conducive to describing (robotic) systems is state-space representations.

<u>state</u> : $x(t) \in \mathbb{R}^n$ (often written compactly as $x$)

state describes the key characteristics of a system.

<u>ex.</u> position, velocity, joint config.

<u>control</u> : $u(t) \in \mathbb{R}^m$

inputs that we choose @ each instance in time.

<u>ex.</u> velocity, motor torques.

<u>output / observation</u> : $y(t) \in \mathbb{R}^l$

outputs that are measurable by our system (through sensor)

<u>ex.</u> speed from speedometer, position via GPS

$\Rightarrow$ for now, $x \equiv y$ assumed but this is often not the case, especially for modern perception - driven robotics.

**System dynamics**: describe how our system evolves over time.

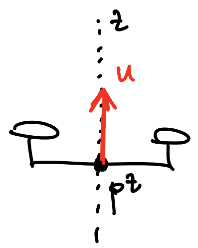
| continuous time | discrete-time |
|---|---|
| $t \in \mathbb{R}$ | $t \in \mathbb{Z}$ |
| $\frac{dx(t)}{dt} = \dot{x} = f(x(t), u(t))$ | $x_{t+1} = f(x_t, u_t)$ |
| common in ctrls, systems theory, safety | MPC, RL, AI ... |

Common: discrete-time approx of continuous-time dyn. sys:

accurate for small $\Delta t$

e.g. forward Euler discretization: $x_{t+1} \approx x_t + \Delta t \cdot f(x, u)$

$\boxed{\text{EXAMPLE}}$ longitudinal quadrotor motion




$$x = \begin{bmatrix} p_z \\ v_z \end{bmatrix}$$

altitude

vertical vel.

$u \in [0, 1]$ normalized thrust

gravity

$$\dot{x} = \begin{bmatrix} v_z \\ k_T u + k_g \end{bmatrix} = \begin{bmatrix} \dot{p}_z \\ \dot{v}_z \end{bmatrix} = f(x(t), u(t))$$

constant. ctrl input

**Trajectory notation**: state, control are functions of time. To make time more explicit, we denote the <u>evolution</u> of a system's state w/ this trajectory notation:

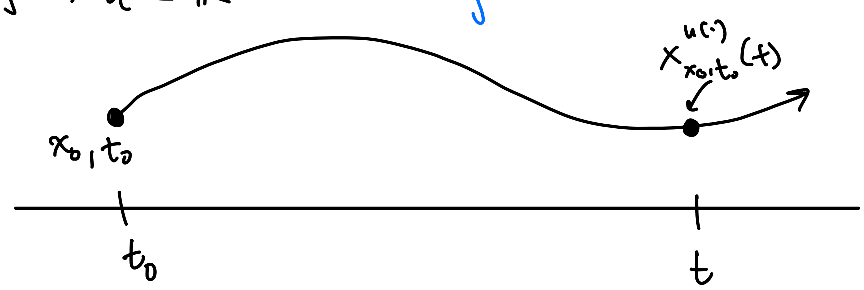
clean, less info. ugly, more info.

$$x = x(t) = x^{u(\cdot)}_{x_0, t_0}(t)$$

"current state" "state @ time t" "state @ time t achieved by system starting @ $x_0$ at time $t_0$ and applying control signal $u(\cdot)$"

$\mathbf{x}(\cdot) : [t_0, T] \to \mathcal{X} \subseteq \mathbb{R}^n$ ← state traj

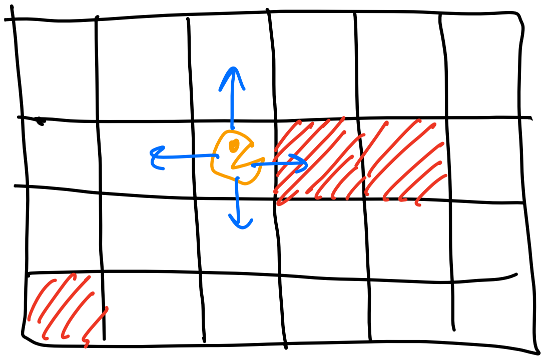
$\mathbf{u}(\cdot) : [t_0, T] \to \mathcal{U} \subseteq \mathbb{R}^m$ ← ctrl. traj



## Need for Safety Analysis: let's discuss why we even need safety analysis.

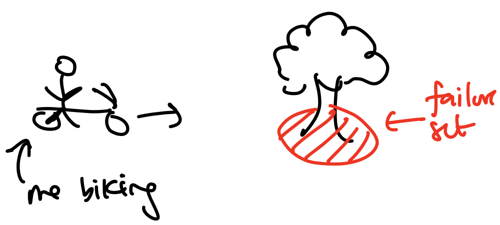
If we know failure set -- e.g. obstacles in a room -- isn't that enough?

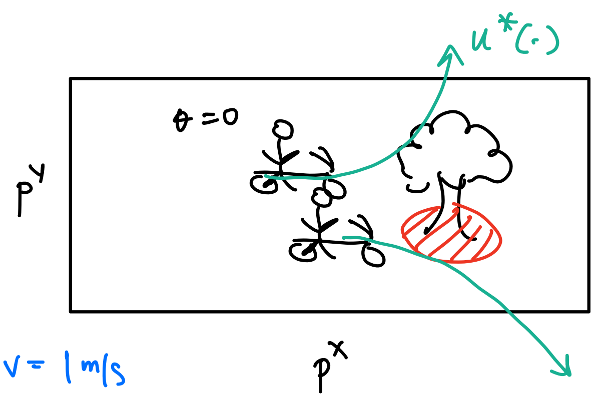
ex. in INTRO TO AI, you see gridworlds where agent moves {UP, DOWN, LEFT, RIGHT} & in this super simple case, just "don't move right" and the agent will be safe!

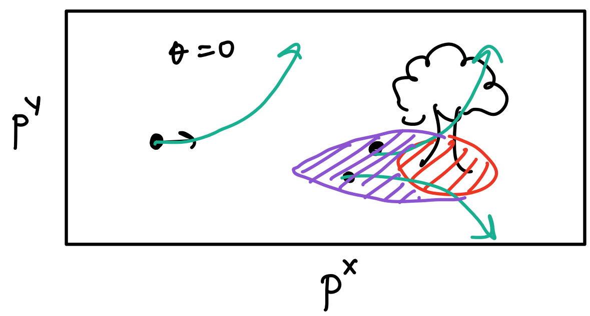


### Reason #1 Inevitable Collisions.

There may be states from which you will reach the failure set even if you are trying your hardest to avoid



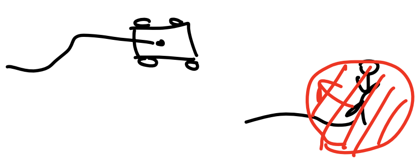




$v = 1$ m/s

$u = \dot{\theta}$ is bounded

### Reason #2 other (strategic) agents

Homicidal chauffer problem (1971)



chauffer is trying to hit the pedestrian. Driver is faster than the pedestrian in lin. speed but is slower to turn. Pedestrian is more maneuverable (directly ctrls. $v^x, v^y$) but is slower. Can the pedestrian stay safe??

## $\boxed{\text{Reason \#3}}$ Uncertainty

Even though we represent the system via a mathematical model, a model will never be perfect in its representation of reality

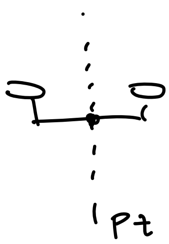
Statistician George Box

"All models are wrong, but some are useful"

simulator, behavior prediction model, physics model, generative model

$$\boxed{\forall \text{ models}, \exists \text{ a reality gap}}$$

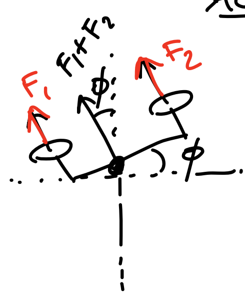
**Simple Model**




$$x = \begin{bmatrix} p_z \\ v_z \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} v_z \\ k_T u + k_g \end{bmatrix}$$

**Advanced model**




$$x = \begin{bmatrix} p_z \\ v_z \\ \phi \\ \omega \end{bmatrix}$$

ctrl input 1

ctrl input 2

STILL NOT PERFECT! $\rightarrow$

$$\dot{x} = \begin{bmatrix} v_z \\ g + k\cos(\phi)(F_1 + F_2) \\ \omega \\ k_\omega(F_1 - F_2) \end{bmatrix}$$

## Uncertainty Representation

Uncertainty in our systems can be classified by it's nature & representation.

For example, uncertainty can be represented non-deterministically or probabilistically

Similarly, it can be structured or unstructured

**Probabilistic**

"I have been observing data & gathering statistics"

- distributions
- transition measures

$$x_{t+1} \sim P(x_{t+1} \mid x_t, u_t)$$

$\leftarrow$ discrete-time $\rightarrow$

**Non-deterministic**

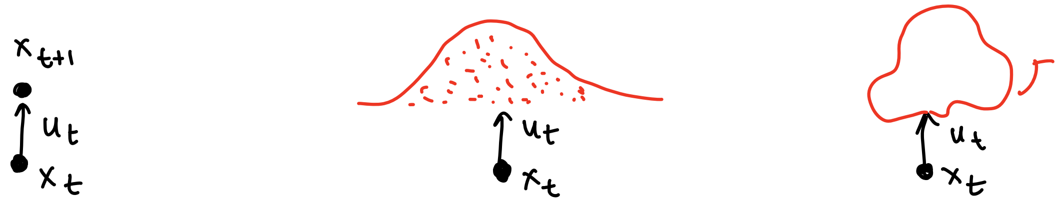
"I have little to no additional information"

- sets
- dynamical inclusion

$$x_{t+1} \in \underbrace{F(x_t, u_t)}_{\text{set}} \subseteq \mathbb{R}^n$$

stochastic differential eqn.

$dX = f(x,u)\,dt + dW$ ← continuous-time → $\dot{x} \in F(x_t, u) \subseteq \mathbb{R}^n$




NO UNCERTAINTY

PROBABILISTIC UNCERTAINTY

NON-DETERMINISTIC UNCERTAINTY

ex. Suppose we are predicting a pedestrian. Model them as a 2D particle moving in the 2D plane

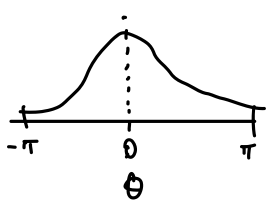
fixed walking speed

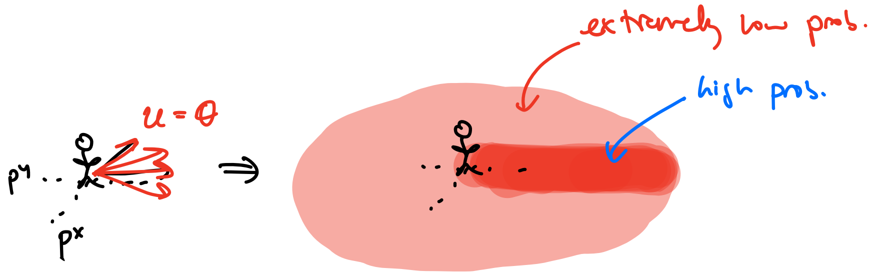
$\theta := u \in [-\pi, \pi]$

$$x = \begin{bmatrix} p^x \\ p^y \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{p}^x \\ \dot{p}^y \end{bmatrix} = \begin{bmatrix} v \cos\theta \\ v \sin\theta \end{bmatrix}$$

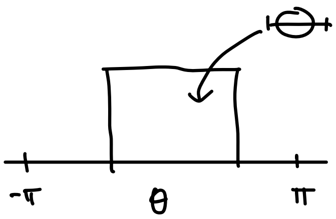
Uncertainty is over $\theta$ (i.e. "$u$" control action)

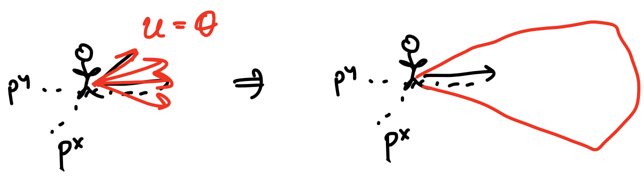
<u>Probabilistic Model</u>




<u>NON-DETERMINISTIC</u>: Assume $\theta \in \Theta$ in some set & give set as output




The safety tools & assurances that you get are diff. btwn. these representations!

# Types of uncertainty:

## Unstructured

- not modelled in a particularly informed way
- often simple to setup, but yields more conservative solns.

ex. $\dot{x} = f(x,u) + d$ ← additive

$d \sim \mathcal{N}(\cdots)$

$d \in \mathcal{D} \subseteq \mathbb{R}^n$

## Structured (parametric)

- enters in a known/structured way
- enters through parameters.

ex. $\dot{x} = f(x,u,d)$

$\dot{p}_z = v_z$

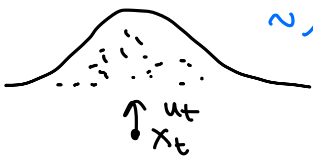
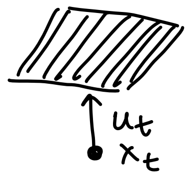
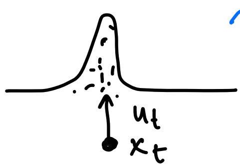
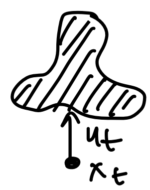
$\dot{v}_z = k_T u + k_g$

$\tau = d$

## Uncertainty Matrix

Probabilistic: assurances: prob.; perf: avg. case

Non-deterministic: assurances: robust; performance: worst case

| | Probabilistic | Non-deterministic |
|---|---|---|
| Unstructured | $x_t = f(x_t,u_t) + d_t$ $\sim \mathcal{N}(\ldots)$  | $x_{t+1} = f(x_t,u_t) + d_t$ $\in \mathcal{D}$  |
| Structured | $x_{t+1} = f(x_t,u_t,d_t)$ $\sim \mathcal{N}(\ldots)$  | $x_{t+1} = f(x_t,u_t,d_t)$ $\in \mathcal{D}$  |